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PROPAGATION IN BI–ISOTROPIC MEDIA: EFFECT OF DIFFERENT FORMALISMS ON THE PROPAGATION ANALYSIS

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1. Introduction

The effect of medium expresses itself in electromagnetics through material parameters in the constitutive relations. There are four vector quantities that appear in Maxwell equations, two field quantities and two flux density quantities, and the connection between these is given in these relations. It is obvious that the most general linear medium needs to be characterized by no less than four dyadics, and indeed four are sufficient. If the medium is isotropic, no direction is prefered, and these parameters are scalars: the medium is bi–isotropic. Four real parameters are enough for lossless media, whereas for lossy bi–isotropic materials, the parameters are complex scalars. Complexity also entails dispersion as the parameters are functions of the frequency of the operating electromagnetic wave.

In addition to permittivity and permeability representing the electric and magnetic polarizability of the material, the other two material parameters stand for the cross–coupling of the electric and magnetic quantities. One physical way of separating these quantities is to decompose the cross–coupling into two parameters that stand for chirality (the degree of the handedness of the medium) and nonreciprocity. With the advent of novel media in microwave and millimeter wave technology, great emphasis has been placed on the theoretical analysis of these chiral and nonreciprocal materials. Due to the diversity of research groups, different approaches in the problem have been taken, starting from the basic description of the media. The literature contains several constitutive relations for bi–isotropic media.

This paper is an attempt to compare different formalisms about the characterization of bi–isotropic media. Five different sets of constitutive relations are presented. Connections are shown between the material parameters, defined within different formalisms. Next the focus will be on wave propagation and Helmholtz equation resolution in bi–isotropic media; a discussion about the mathematical and physical aspects of different formalisms is given in the last chapter.

2. Constitutive Equations for Bi-isotropic Media

The constitutive relations that are needed to fully describe general bi–isotropic media require four scalar material parameters. There are different constitutive equations due to the various possibilities to link the electric (**E** and **D**) and the magnetic (**H** and **B**) field and flux quantities.

In the present analysis, we assume sinusoidal time dependence with the convention: $e^{j\omega t}$.

2.1 A and B Formalisms

These relations have been first given by Lindell and Sihvola [1]. The constitutive equations are written as follows:

$$\begin{aligned} \mathbf{D} &= \varepsilon_A \mathbf{E} + (\chi - j\kappa)\sqrt{\varepsilon_o \mu_o}\,\mathbf{H} \\ \mathbf{B} &= \mu_A \mathbf{H} + (\chi + j\kappa)\sqrt{\varepsilon_o \mu_o}\,\mathbf{E} \end{aligned} \tag{1}$$

κ is the chirality parameter (dimensionless), χ is the nonreciprocity parameter (also dimensionless).

Condon [2] described chiral media with the characteristic parameter χ_B and Tellegen [3] included the nonreciprocity effect in the quantity γ_B. These two formalisms can be combined and the resulting constitutive equations look like:

$$\begin{aligned} \mathbf{D} &= \varepsilon_B \mathbf{E} + \gamma_B \mathbf{H} - \chi_B \frac{\partial \mathbf{H}}{\partial t} \\ \mathbf{B} &= \mu_B \mathbf{B} + \gamma_B \mathbf{E} + \chi_B \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{2}$$

or

$$\begin{aligned} \mathbf{D} &= \varepsilon_B \mathbf{E} + (\gamma_B - j\omega\chi_B)\mathbf{H} \\ \mathbf{B} &= \mu_B \mathbf{H} + (\gamma_B + j\omega\chi_B)\mathbf{E} \end{aligned} \tag{3}$$

γ_B is the nonreciprocity parameter $(s \cdot m^{-1})$, and χ_B is the chirality parameter $(s^2 \cdot m^{-1})$. These equations are named formalism B. Crossing equations between Lindell Sihvola and Condon Tellegen formalisms are then quite obvious and are listed below:

$$\begin{aligned} \varepsilon_B &= \varepsilon_A \\ \mu_B &= \mu_A \\ \gamma_B &= \chi \ \sqrt{\varepsilon_o \mu_o} \\ \chi_B &= \frac{\kappa \sqrt{\varepsilon_o \mu_o}}{\omega} \end{aligned} \tag{4}$$

2.2 Formalism C

In 1988, Engheta and Jaggard [4,5] used others constitutive equations, linking **D** and **H** to, **E** and **B**, for reciprocal media. Then these equations have been extended to the case of bi–isotropic media [9].

The constitutive equations are written as follows:

$$\begin{aligned} \mathbf{D} &= \varepsilon_c \mathbf{E} + (\Psi_n - j\xi_c)\mathbf{B} \\ \mathbf{H} &= \frac{\mathbf{B}}{\mu_c} - (\Psi_n + j\xi_c)\mathbf{E} \end{aligned} \tag{5}$$

Ψ_n and ξ_c are, respectively, nonreciprocity and chirality parameters; they represent admittances.

Crossing equations between formalisms A and C are given by:

$$\begin{aligned}
\varepsilon_c =& \varepsilon_A - \frac{\mu_o \varepsilon_o}{\mu_A}\left(\chi^2 + \kappa^2\right) \\
\mu_c =& \mu_A \\
\Psi_n =& \frac{\chi \sqrt{\varepsilon_o \mu_o}}{\mu_A} \\
\xi_c =& \frac{\kappa \sqrt{\varepsilon_o \mu_o}}{\mu_A}
\end{aligned} \tag{6}$$

2.3 Formalism D

This set of relations has been used by Lakhtakia and Varadan [6] to characterize reciprocal chiral media. It may also be attributed to Drude, Born, and Fedorov [7].

A trial to generalize the constitutive relations for general bi–isotropic media was made, through the equations:

$$\begin{aligned}
\mathbf{D} =& \varepsilon_D \left(\mathbf{E} + (\beta + j\alpha)\mathbf{rot}\,\mathbf{E}\right) \\
\mathbf{B} =& \mu_D \left(\mathbf{H} + (\beta - j\alpha)\mathbf{rot}\,\mathbf{H}\right)
\end{aligned} \tag{7}$$

α and β are, respectively, the nonreciprocity and the chirality parameters. Their dimension is that of length.

Using Maxwell equations, (7) can be written as follows:

$$\begin{aligned}
\mathbf{E} =& \frac{\mathbf{D}}{\varepsilon_D} + j\omega(\beta + j\alpha)\mathbf{B} \\
\mathbf{H} =& \frac{\mathbf{B}}{\mu_D} - j\omega(\beta - j\alpha)\mathbf{D}
\end{aligned} \tag{8}$$

Crossing equations between Lindell Sihvola and Drude Born Fedorov formalisms are given by:

$$
\begin{aligned}
\varepsilon_D =& \varepsilon_A \left(1 - \frac{\varepsilon_o \mu_o}{\varepsilon_A \mu_A} \left(\chi^2 + \kappa^2\right)\right) \\
\mu_D =& \mu_A \left(1 - \frac{\varepsilon_o \mu_o}{\varepsilon_A \mu_A} \left(\chi^2 + \kappa^2\right)\right) \\
\alpha =& \frac{\chi \sqrt{\varepsilon_o \mu_o}}{\omega \left(\varepsilon_A \mu_A - \left(\chi^2 + \kappa^2\right) \varepsilon_o \mu_o\right)} \\
\beta =& \frac{\kappa \sqrt{\varepsilon_o \mu_o}}{\omega \left(\varepsilon_A \mu_A - \left(\chi^2 + \kappa^2\right) \varepsilon_o \mu_o\right)}
\end{aligned}
\tag{9}
$$

2.4 Formalism E

A further possibility to link the flux and field quantities is to connect $(\mathbf{E}, \mathbf{B})$ with $(\mathbf{D}, \mathbf{H})$. In this case, constitutive equations are written as follows:

$$
\begin{aligned}
\mathbf{E} =& \frac{\mathbf{D}}{\varepsilon_E} - (\alpha_1 - j\alpha_2)\mathbf{H} \\
\mathbf{B} =& \mu_E \mathbf{H} + (\alpha_1 + j\alpha_2)\mathbf{D}
\end{aligned}
\tag{10}
$$

α_2 and α_1 are the chirality and nonreciprocity parameters. They represent impedances.

Crossing equations between formalisms E and A are the following ones:

$$
\begin{aligned}
\varepsilon_E =& \varepsilon_A \\
\mu_E =& \mu_A - \left(\frac{\chi^2 \varepsilon_o \mu_o}{\varepsilon_A} + \frac{\kappa^2 \varepsilon_o \mu_o}{\varepsilon_A}\right) \\
\alpha_1 =& \frac{\chi \sqrt{\varepsilon_o \mu_o}}{\varepsilon_A} \\
\alpha_2 =& \frac{\kappa \sqrt{\varepsilon_o \mu_o}}{\varepsilon_A}
\end{aligned}
\tag{11}
$$

It is worth noting that in any formalism, the four material parameters (ε , μ, chirality and nonreciprocity parameters) are complex numbers for a lossy medium.

The constitutive relations for isotropic media can be deduced from all these formalisms by setting the chirality and nonreciprocity parameters to zero.

3. Propagation in Bi-isotropic Media

Using both Maxwell equations and constitutive relations for a bi-isotropic medium, we can deduce the propagation equation, which is given for each formalism in Table 1.

	propagation equation
A	$\mathbf{Rot\,Rot\,E} - 2\kappa\omega\sqrt{\varepsilon_o\mu_o}\,\mathbf{Rot\,E} - \omega^2\left(\varepsilon_A\mu_A - \varepsilon_o\mu_o\left(\chi^2+\kappa^2\right)\right)\mathbf{E} = 0$
B	$\mathbf{Rot\,Rot\,E} - 2\chi_B\omega^2\,\mathbf{Rot\,E} - \omega^2\left(\varepsilon_B\mu_B - \left(\gamma_B^2+\omega^2\chi_B^2\right)\right)\mathbf{E} = 0$
C	$\mathbf{Rot\,Rot\,E} - 2\xi_c\mu_C\omega\,\mathbf{Rot\,E} - \omega^2\varepsilon_C\mu_C\mathbf{E} = 0$
D	$\mathbf{Rot\,Rot\,E} - \dfrac{2\beta\varepsilon_D\mu_D\omega^2}{1-\omega^2\varepsilon_D\mu_D\left(\alpha^2+\beta^2\right)}\,\mathbf{Rot\,E} - \dfrac{\omega^2\varepsilon_D\mu_D}{1-\omega^2\varepsilon_D\mu_D\left(\alpha^2+\beta^2\right)}\mathbf{E} = 0$
E	$\mathbf{Rot\,Rot\,E} - 2\alpha_2\varepsilon_E\omega\,\mathbf{Rot\,E} - \omega^2\varepsilon_E\mu_E\mathbf{E} = 0$

Table 1.

This equation can be easily solved, by the means of Bohren decomposition [8], which replaces the unknown fields E and H by two left and right circular polarized waves (LCP and RCP), defined as follows:

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = [A] \begin{bmatrix} \mathbf{Q}_{\mathrm{LCP}} \\ \mathbf{Q}_{\mathrm{RCP}} \end{bmatrix}$$

These two waves are both solutions of the homogeneous Helmholtz propagation equation:

$$\begin{aligned} \Delta \mathbf{Q}_{\mathrm{LCP}} + k_{\mathrm{LCP}}^2 \mathbf{Q}_{\mathrm{LCP}} &= 0 \\ \Delta \mathbf{Q}_{\mathrm{RCP}} + k_{\mathrm{RCP}}^2 \mathbf{Q}_{\mathrm{RCP}} &= 0 \end{aligned}$$

The two LCP and RCP propagation constants are given in Table 2 for the various formalisms. The difference between the real parts of the RCP and LCP numbers induces polarization rotation, while the difference between the imaginary parts is related to circular dichroism.

Resolution of these equations leads to the following results:

$$A = \begin{bmatrix} 1 & -jZ_{\mathrm{RCP}} \\ -j/Z_{\mathrm{LCP}} & 1 \end{bmatrix}$$

where Z is defined as the wave impedance and then

$$\begin{aligned} \mathbf{E} &= \mathbf{Q}_{\mathrm{LCP}} - jZ_{\mathrm{RCP}} \mathbf{Q}_{\mathrm{RCP}} \\ \mathbf{H} &= -jZ_{\mathrm{LCP}}^{-1} \mathbf{Q}_{\mathrm{LCP}} + \mathbf{Q}_{\mathrm{RCP}} \end{aligned}$$

Wave impedances within the various formalisms are reviewed in Table 3.

	propagation constants of the RCP and LCP waves
A	$k_{\mathrm{RCP}} = \omega\left(\kappa\sqrt{\varepsilon_o\mu_o} + \sqrt{\varepsilon_A\mu_A - \varepsilon_o\mu_o\chi^2}\right)$ $k_{\mathrm{LCP}} = \omega\left(-\kappa\sqrt{\varepsilon_o\mu_o} + \sqrt{\varepsilon_A\mu_A - \varepsilon_o\mu_o\chi^2}\right)$
B	$k_{\mathrm{RCP}} = \omega\left(\omega\chi_B + \sqrt{\varepsilon_B\mu_B - \gamma_B^2}\right)$ $k_{\mathrm{LCP}} = \omega\left(-\omega\chi_B + \sqrt{\varepsilon_B\mu_B - \gamma_B^2}\right)$
C	$k_{\mathrm{RCP}} = \omega\left(\mu_C\xi_c + \sqrt{\varepsilon_C\mu_C + \mu_C^2\xi_c^2}\right)$ $k_{\mathrm{LCP}} = \omega\left(-\mu_C\xi_c + \sqrt{\varepsilon_C\mu_C + \mu_C^2\xi_c^2}\right)$
D	$k_{\mathrm{RCP}} = \omega\left(\frac{-\omega\mu_D\varepsilon_D\beta + \sqrt{\varepsilon_D\mu_D - \omega^2\mu_D\varepsilon_D\alpha^2}}{1-\omega^2\varepsilon_D\mu_D\left(\alpha^2+\beta^2\right)}\right)$ $k_{\mathrm{LCP}} = \omega\left(\frac{\omega\mu_D\varepsilon_D\beta + \sqrt{\varepsilon_D\mu_D - \omega^2\mu_D\varepsilon_D\alpha^2}}{1-\omega^2\varepsilon_D\mu_D\left(\alpha^2+\beta^2\right)}\right)$
E	$k_{\mathrm{RCP}} = \omega\left(\varepsilon_E\alpha_2 + \sqrt{\varepsilon_E\mu_E + \varepsilon_E^2\alpha_2^2}\right)$ $k_{\mathrm{LCP}} = \omega\left(-\varepsilon_E\alpha_2 + \sqrt{\varepsilon_E\mu_E + \varepsilon_E^2\alpha_2^2}\right)$

Table 2.

	wave impedances
A	$Z_{\substack{\text{LCP}\\\text{RCP}}} = \dfrac{\mu_A}{\sqrt{\varepsilon_A \mu_A - \varepsilon_o \mu_o \chi^2} \pm j\chi\sqrt{\varepsilon_o \mu_o}}$
B	$Z_{\substack{\text{LCP}\\\text{RCP}}} = \dfrac{\mu_B}{\sqrt{\varepsilon_B \mu_B - \gamma_B^2} \pm j\gamma_B}$
C	$Z_{\substack{\text{LCP}\\\text{RCP}}} = \dfrac{\mu_C}{\sqrt{\varepsilon_C \mu_C + \mu_C^2 \xi_C^2} \pm j\Psi_n \mu_C}$
D	$Z_{\substack{\text{LCP}\\\text{RCP}}} = \dfrac{\mu_D}{\sqrt{\varepsilon_D \mu_D - \omega^2 \alpha^2 \varepsilon_D^2 \mu_D^2} \pm j\omega\varepsilon_D \mu_D \alpha}$
E	$Z_{\substack{\text{LCP}\\\text{RCP}}} = \dfrac{\mu_E + \varepsilon_E\left(\alpha_1^2 + \alpha_2^2\right)}{\sqrt{\varepsilon_E \mu_E + \varepsilon_E^2 \alpha_2^2} \pm j\varepsilon_E \alpha_1}$

Table 3.

In each table, well known equations for reciprocal and achiral media can be refound by equating both nonreciprocity and chirality parameters to zero.

4. Definition of the Pair (ε, μ) in a Bi-isotropic Media

Because of the different ways of including electromagnetic coupling in the constitutive relations, the permittivity ε and the permeability μ differ in various formalisms . Let us compare these definitions, referring to the convention proposed by Kong [9], where ε and μ are the diagonal terms of the matrix connecting $(\mathbf{D}, \mathbf{B})$ with $(\mathbf{E}, \mathbf{H})$.

The A and B formalisms agree with the Kong's definition (i.e., ε and μ are not changed). The other formalisms have their own definition. This problem is not only formal: physical interpretations may also depend on the choice of the formalism.

For example, in the A and B formalisms, the propagation equation, and then the wave numbers, are function of the nonreciprocity parameters. On the contrary, they are independent of this parameter in the C and E formalisms.

Elsewhere, in the C and E formalisms, the wave impedance is a function of the chirality parameter, while in the A and B formalisms, the wave impedance is independent of this parameter.

In the D formalism, the propagation equation, the wave numbers and the wave impedance are both dependent on chirality and nonreciprocity parameters.

In fact, a bi–isotropic medium is completely characterized by a particular set of four parameters (permittivity, permeability, chirality, and nonreciprocity), in any formalism. And, due to the mathematical equivalence between all the formalisms, all of them will calculate the same numerical value for the wavenumbers or the wave impedance, although each set of four material parameters is changed between the various formalisms. In practice, there will be no ambiguity if any numerical value of material parameters or analytical expression in bi–isotropic media is clearly said to be found in the the A B, C, D, or E formalism.

From an energy conservation law point of view, the A, B, and C formalisms show good agreement whatever the material parameters are; in the D formalism, equations (8) have to be used for energy conservation law in any bi–isotropic medium. Other elements about energetic considerations for bi–isotropic media are given in [10], published in this issue.

5. Conclusion

Several formalisms can be applied to the electromagnetic analysis of bi–isotropic media.

After reviewing the different ways of coupling electromagnetic fields $\mathbf{D}, \mathbf{B}, \mathbf{E}$, and $\mathbf{H}$, mathematical equivalence between them has been demonstrated, and the crosslinks to pass from one formalism to another have been established.

It appears that there is no best set of equations. For each problem, one set will turn out to be simpler than the others, and everywhere in the calculations, it is possible to pass to another formalism.

However, a physical interpretation must be limited to the formalism where it has been established; this apparent paradox is to be related to thé particular definitions of ε , μ, chirality and nonreciprocity parameters in each formalism. The same values will be computed for the basic quantities (wave impedance and wave numbers for LCP and RCP waves) within any formalism.

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