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POLARIZABILITY MATRIX OF LAYERED CHIRAL SPHERE

M. E. Ermutlu and A. H. Sihvola

1. Introduction

Novel material effects are one of today's interesting fields in the study of electromagnetic fields and waves, not only for academic reasons but also because of the promise they seem to offer in microwave technology applications. Chiral media certainly form a class of these complex materials, since the number of publications about chiral medium in microwave and millimeter wave theory and techniques of the latest years are continuously increasing. Chirality is present in handed — left–right nonsymmetric — materials. As far as constitute equations are concerned this handedness is visible in the magnetoelectric coupling

$$\mathbf{D} = \epsilon\mathbf{E} - j\kappa\sqrt{\mu_0\epsilon_0}\,\mathbf{H} \tag{1}$$

$$\mathbf{B} = \mu\mathbf{H} + j\kappa\sqrt{\mu_0\epsilon_0}\,\mathbf{E} \tag{2}$$

where $\mathbf{E}$ is the electric and $\mathbf{H}$ is the magnetic field, $\mathbf{D}$ is the electric and $\mathbf{B}$ is the magnetic flux density. In addition to the permittivity ϵ

and permeability μ of the material, there exists a dimensionless parameter κ which is a measure of the chirality, or handedness [1]. These relations assume time-harmonic field dependence with the convention $\exp(j\omega t)$. The permittivity and permeability of free space are ϵ_0 and μ_0.

Several of the classical electromagnetic free-space and boundary-value problems that involve chiral materials have been solved. The problem of scattering by electrically small chiral objects has also received attention [2,3]. The polarizability matrices of homogeneous chiral spheres and ellipsoids have been computed. This article focuses on a more complicated problem: the inhomogeneous chiral sphere. The inhomogeneity is, however, simple: the object consists of a spherical chiral core, coated with a spherical shell of a different chiral material. The polarizability properties of this object will be solved.

The polarizability matrix is related to the induced electric and magnetic dipole moments to the incident electric and magnetic fields. Due to the chirality parameters, the sphere will become magnetically polarized by an electric field [1] (without any incident magnetic field), and vice versa. However, the polarizability coefficients are complicated functions of the various parameters of the problem.

Following the analysis performed earlier on an homogeneous chiral sphere [4], where the potential is expanded in spherical harmonics inside and outside the sphere, this article adds new unknowns and makes use of extra boundary conditions that are present in the problem of the layered sphere. It is worth noting that the sphere is assumed to be small compared to the wavelength, i.e. Rayleigh scattering is assumed. This assumption leads to fairly simple potential solutions in each region. [2] However, for the case of two spherical boundaries, it turns out that the boundary conditions are a set of eight coupled equations that is written in matrix form. Solving this set requires — or at least will be made a lot easier by — symbolic software.

With known symbolic calculation softwares like MACSYMA, MATHEMATICA, or MAPLE, the solution of the problem can be found easily. However the analytic expressions are long, complicated, and dependent on several variables so that computer programs are not capable of simplifying the expressions without human interaction.

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[1] This is in addition to the ordinary electric polarization.

[2] For electrically large chiral spheres, the generalized Mie scattering leads to complicated series expansions [5].

After the matrix has been inverted, the scattered fields can be studied. Interpreting the latter as arising from electric and magnetic dipoles, which are directly proportional to the incident electric and magnetic fields due to the linearity of the problem, the polarizability matrix can be written. The result can be shown to reduce to the previously known cases of homogeneous chiral and layered dielectric sphere, corroborating the result.

2. Theory

2.1 Wave Fields

In the study of electromagnetic fields in homogeneous chiral media, it is advantageous to use so called "wave fields" [6] instead of the ordinary electric and magnetic fields. These are defined through equations

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{3}$$

$$\mathbf{H} = -\frac{1}{j\eta}(\mathbf{E}_+ - \mathbf{E}_-) \tag{4}$$

where $\eta = \sqrt{\mu/\epsilon}$.

The wave fields $\mathbf{E}_+, \mathbf{E}_-$ decouple as they are substituted in Maxwell equations and behave like in isotropic media; the $+$ and $-$ fields propagate through the chiral medium with different wave numbers. In source-free regions, the wave fields satisfy the first-order partial differential equations

$$\nabla \times \mathbf{E}_\pm \mp k_\pm \mathbf{E}_\pm = 0 \tag{5}$$

with $k_\pm = \omega\sqrt{\mu_0\epsilon_0}(n \pm \kappa)$, and the refractive index is $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$.

2.2 Quasi-Static Case

Following the procedure used in [4] for solving the quasi-static potential inside a chiral sphere, the chiral layered sphere is studied in a similar form. In the static limit $k_\pm \to 0$, the wave fields can be obtained from scalar potentials $\phi_\pm$:

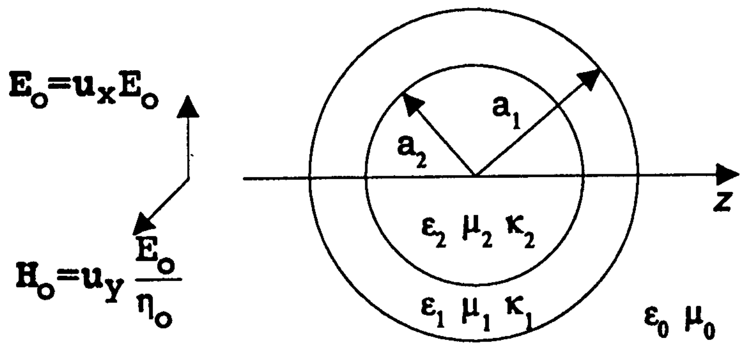

Figure 1. Geometry of the problem. Two-layer chiral sphere with outer and inner radii are a_1 and a_2, respectively in air with ϵ_0, μ_0. The geometry consists of a spherical core with parameters $\epsilon_2, \mu_2, \kappa_2$, surrounded by spherical shell with $\epsilon_1, \mu_1, \kappa_1$.

$$\mathbf{E}_\pm = -\nabla\phi_\pm \tag{6}$$

satisfying the Laplace equation

$$\nabla^2\phi_\pm = 0. \tag{7}$$

The question is to find the internal and external fields of a two-layer chiral sphere with outer and inner sphere radii a_1, a_2, respectively. Let the sphere be surrounded by vacuum as is shown in Figure 1. Let us denote the incident field by the subscript o, scattered fields by s and internal fields by $i1$ and $i2$. This leads to 8 potentials that satisfy the Laplace equation.

In the following, the material parameters, permitivity ϵ, permeability μ, chirality κ, refractive index $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$, and the impedance $\eta = \sqrt{\mu/\epsilon}$ have the subscripts 0 for the air region, 1 for the layer, and 2 for the core. There are four interface conditions for the total fields ($\mathbf{E}, \mathbf{H}$) and total fluxes ($\mathbf{D}, \mathbf{B}$). These conditions give the following equations for the unknown potentials, using equations (3), (4), and (6):

- **Continuity of the total tangential electric fields at the interfaces:**

$$\phi_{i1+} + \phi_{i1-} = \phi_{s+} + \phi_{s-} + \phi_{o+} + \phi_{o-} \quad | \, r = a_1 \tag{8}$$

$$\phi_{i1+} + \phi_{i1-} = \phi_{i2+} + \phi_{i2-} \quad | \, r = a_2 \tag{9}$$

- **Continuity of the total tangential magnetic field:**

$$\frac{1}{\eta_1}(\phi_{i1+} - \phi_{i1-}) = \frac{1}{\eta_o}(\phi_{s+} - \phi_{s-} + \phi_{o+} - \phi_{o-}) \tag{10}$$

$$\frac{1}{\eta_1}(\phi_{i1+} - \phi_{i1-}) = \frac{1}{\eta_2}(\phi_{i2+} - \phi_{i2-}) \tag{11}$$

- **Continuity of the normal component of the total electric flux density vector:**
  Using the relation

$$\mathbf{D} = \epsilon\mathbf{E} - j\kappa\sqrt{\mu_0\epsilon_0}\,\mathbf{H} = \epsilon(1 + \frac{\kappa}{n})\mathbf{E}_+ + \epsilon(1 - \frac{\kappa}{n})\mathbf{E}_- \tag{12}$$

we have

$$\epsilon_1(1 + \frac{\kappa_1}{n_1})\mathbf{u}_r \cdot \nabla\phi_{i1+} + \epsilon_1(1 - \frac{\kappa_1}{n_1})\mathbf{u}_r \cdot \nabla\phi_{i1-} =$$

$$\epsilon_0[\mathbf{u}_r \cdot \nabla\phi_{s+} + \mathbf{u}_r \cdot \nabla\phi_{s-} + \mathbf{u}_r \cdot \nabla\phi_{o+} + \mathbf{u}_r \cdot \nabla\phi_{o-}] \quad | \, r = a_1 \tag{13}$$

$$\epsilon_1(1 + \frac{\kappa_1}{n_1})\mathbf{u}_r \cdot \nabla\phi_{i1+} + \epsilon_1(1 - \frac{\kappa_1}{n_1})\mathbf{u}_r \cdot \nabla\phi_{i1-} =$$

$$\epsilon_2(1 + \frac{\kappa_2}{n_2})\mathbf{u}_r \cdot \nabla\phi_{i2+} + \epsilon_1(1 - \frac{\kappa_2}{n_2})\mathbf{u}_r \cdot \nabla\phi_{i2-} \quad | \, r = a_2 \tag{14}$$

- **4. Continuity of the normal component of the magnetic flux density vector:**
  Using the relation

$$\mathbf{B} = \mu\mathbf{H} + j\kappa\sqrt{\mu_0\epsilon_0}\,\mathbf{E} = j\sqrt{\mu_0\epsilon_0}[(n + \kappa)\mathbf{E}_+ - (n - \kappa)\mathbf{E}_-] \tag{15}$$

we have

$$(n_1+\kappa_1)\mathbf{u}_r\cdot\nabla\phi_{i1+}-(n_1-\kappa_1)\mathbf{u}_r\cdot\nabla\phi_{i1-}=$$
$$\mathbf{u}_r\cdot\nabla\phi_{s+}-\mathbf{u}_r\cdot\nabla\phi_{s-}+\mathbf{u}_r\cdot\nabla\phi_{o+}-\mathbf{u}_r\cdot\nabla\phi_{o-}\quad|\, r=a_1 \tag{16}$$

$$(n_1+\kappa_1)\mathbf{u}_r\cdot\nabla\phi_{i1+}-(n_1-\kappa_1)\mathbf{u}_r\cdot\nabla\phi_{i1-}=$$
$$(n_2+\kappa_2)\mathbf{u}_r\cdot\nabla\phi_{i2+}-(n_2-\kappa_2)\mathbf{u}_r\cdot\nabla\phi_{i2-}\quad|\, r=a_2 \tag{17}$$

Let us write all potentials as spherical functions expansions

$$\phi(\mathbf{r})=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(A_{mn}r^n+B_{mn}r^{-n-1})P_n^m(\cos\theta)e^{jm\varphi} \tag{18}$$

which satisfy the Laplace equation. When the sphere is located in a constant field with electric and magnetic components taken from a "frozen" plane wave,

$$\mathbf{E}_o=\mathbf{u}_x E_o \tag{19}$$

$$\mathbf{H}_o=\mathbf{u}_y\frac{E_o}{\eta_o}, \tag{20}$$

the corresponding potentials are

$$\phi_{o\pm}=-\frac{E_o}{2}(x\mp jy)=\frac{E_o}{2}rP_1^1(\cos\theta)e^{\mp j\varphi}. \tag{21}$$

From the orthogonality of the associated Legendre functions [7] and the exponential functions, it follows that only terms with $n=1$ remain in the above series. The fields corresponding to $\exp(-j\varphi)$ and $\exp(j\varphi)$ must satisfy separate equations. The potential inside the interior sphere does not include the "scattered" term with the r^{-2} dependence since it becomes infinite at the origin. Similarly, the scattered potential does not have a constant term (r dependence) because this does not vanish in the infinity. Therefore the wave-field potentials can be written in the following form,

$$\phi_{i1\pm}=\left([A_{1\pm}e^{-j\varphi}+C_{1\pm}e^{j\varphi}]r+[B_{1\pm}e^{-j\varphi}+D_{1\pm}e^{j\varphi}]r^{-2}\right)P_1^1(\cos\theta) \tag{22}$$

$$\phi_{i2\pm} = [A_{2\pm}e^{-j\varphi} + C_{2\pm}e^{j\varphi}]rP_1^1(\cos\theta) \tag{23}$$

$$\phi_{s\pm} = [B_{2\pm}e^{-j\varphi} + D_{2\pm}e^{j\varphi}]r^{-2}P_1^1(\cos\theta) \tag{24}$$

with $P_1^1(\cos\theta) = -\sin\theta$.

2.3 Matrix Equation for the Unknown Coefficients

Applying all the boundary conditions (8) $\cdots$ (17), and writing separately the $\exp(j\varphi)$ and $\exp(-j\varphi)$ terms, which correspond to the left- and right- circularly polarized components of the incoming field $\mathbf{E}_o$, two matrix equations for the 16 coefficients result. The first one is:

$$[M]\begin{pmatrix} \sqrt{\epsilon_{r1}}A_{1+} \\ \sqrt{\epsilon_{r1}}A_{1-} \\ a_1^{-3}B_{1+} \\ a_1^{-3}B_{1-} \\ \sqrt{\epsilon_{r1}}A_{2+} \\ \sqrt{\epsilon_{r1}}A_{2-} \\ a_1^{-3}B_{2+} \\ a_1^{-3}B_{2-} \end{pmatrix} = \frac{E_o}{2}\begin{pmatrix} \sqrt{\epsilon_{r1}} \\ 0 \\ \sqrt{\mu_{r1}} \\ 0 \\ \sqrt{\mu_{r1}} \\ 0 \\ \sqrt{\epsilon_{r1}} \\ 0 \end{pmatrix} \tag{25}$$

where $[M]$ is 8×8 matrix,

$$\begin{bmatrix} 1 & 1 & \sqrt{\epsilon_{r1}} & \sqrt{\epsilon_{r1}} \\ 1 & 1 & \delta\sqrt{\epsilon_{r1}} & \delta\sqrt{\epsilon_{r1}} \\ 1 & -1 & \sqrt{\epsilon_{r1}} & -\sqrt{\epsilon_{r1}} \\ 1 & -1 & \delta\sqrt{\epsilon_{r1}} & -\delta\sqrt{\epsilon_{r1}} \\ n_1+\kappa_1 & n_1-\kappa_1 & -2(n_1+\kappa_1)\sqrt{\epsilon_{r1}} & -2(n_1-\kappa_1)\sqrt{\epsilon_{r1}} \\ n_1+\kappa_1 & n_1-\kappa_1 & -2\delta(n_1+\kappa_1)\sqrt{\epsilon_{r1}} & -2\delta(n_1-\kappa_1)\sqrt{\epsilon_{r1}} \\ n_1+\kappa_1 & -(n_1-\kappa_1) & -2(n_1+\kappa_1)\sqrt{\epsilon_{r1}} & 2(n_1-\kappa_1)\sqrt{\epsilon_{r1}} \\ n_1+\kappa_1 & -(n_1-\kappa_1) & -2\delta(n_1+\kappa_1)\sqrt{\epsilon_{r1}} & 2\delta(n_1-\kappa_1)\sqrt{\epsilon_{r1}} \end{matrix}$$

$$\begin{matrix} 0 & 0 & -\sqrt{\epsilon_{r1}} & -\sqrt{\epsilon_{r1}} \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -\sqrt{\mu_{r1}} & \sqrt{\mu_{r1}} \\ -\eta_r & \eta_r & 0 & 0 \\ 0 & 0 & 2\sqrt{\mu_{r1}} & 2\sqrt{\mu_{r1}} \\ -\eta_r(n_2+\kappa_2) & -\eta_r(n_2-\kappa_2) & 0 & 0 \\ 0 & 0 & 2\sqrt{\epsilon_{r1}} & -2\sqrt{\epsilon_{r1}} \\ -(n_2+\kappa_2) & n_2-\kappa_2 & 0 & 0 \end{matrix} \Bigg] \tag{26}$$

and where $\delta = a_1^3 a_2^{-3}$ and $\eta_r = \eta_1 \eta_2^{-1}$. Note the relative permittivities $\epsilon_{ri} = \epsilon_i/\epsilon_0$ and relative permeabilities $\mu_{ri} = \mu_i/\mu_0$.

The second one is

$$[M] \begin{pmatrix} \sqrt{\epsilon_{r1}}C_{1+} \\ \sqrt{\epsilon_{r1}}C_{1-} \\ a_1^{-3}D_{1+} \\ a_1^{-3}D_{1-} \\ \sqrt{\epsilon_{r1}}C_{2+} \\ \sqrt{\epsilon_{r1}}C_{2-} \\ a_1^{-3}D_{2+} \\ a_1^{-3}D_{2-} \end{pmatrix} = \frac{E_o}{2} \begin{pmatrix} \sqrt{\epsilon_{r1}} \\ 0 \\ -\sqrt{\mu_{r1}} \\ 0 \\ \sqrt{\mu_{r1}} \\ 0 \\ -\sqrt{\epsilon_{r1}} \\ 0 \end{pmatrix}. \tag{27}$$

The 8×8 matrix M is the same for both equations and has the determinant which is solved using MATHEMATICA.

$$\det[M] = \frac{16 n_1^2 \eta_0}{\eta_2} \Delta \tag{28}$$

where

$$\Delta = \left(\left[(\mu_{r1}+2)(\epsilon_{r1}+2)-\kappa_1^2\right]\delta + 2\left[(\mu_{r1}-1)(\epsilon_{r1}-1)-\kappa_1^2\right]\right)$$
$$\cdot\left(\left[(\mu_{r2}+2\mu_{r1})(\epsilon_{r2}+2\epsilon_{r1})-(\kappa_2+2\kappa_1)^2\right]\delta\right.$$
$$\left.+\,2\left[(\mu_{r2}-\mu_{r1})(\epsilon_{r2}-\epsilon_{r1})-(\kappa_2-\kappa_1)^2\right]\right)$$
$$-\,18\delta\left[(\epsilon_{r1}\mu_{r1}-\kappa_1^2-\epsilon_{r2})(\epsilon_{r1}\mu_{r1}-\kappa_1^2-\mu_{r2})-\kappa_2^2\right] \qquad (29)$$

Since the aim of this paper is to show the polarization moments, only the coefficients necessary for the derivation of these will be calculated.

2.4 Polarization Moments

The scattered fields can be regarded as radiated by a combination of electric and magnetic dipole sources. Since

$$\phi_{s\pm}(\mathbf{r}) = \left(B_{2\pm}e^{-j\varphi} + D_{2\pm}e^{j\varphi}\right) r^{-2} P_1^1(\cos\theta), \qquad (30)$$

the wave-field dipole moments can be written as

$$\mathbf{p}_+ = -4\pi\epsilon_0\left[(B_{2+}+D_{2+})\mathbf{u}_x - j(B_{2+}-D_{2+})\mathbf{u}_y\right] \qquad (31)$$

and

$$\mathbf{p}_- = -4\pi\epsilon_0\left[(B_{2-}+D_{2-})\mathbf{u}_x - j(B_{2-}-D_{2-})\mathbf{u}_y\right]. \qquad (32)$$

Because [4]

$$\mathbf{p}_e = \mathbf{p}_+ + \mathbf{p}_-, \qquad (33)$$

$$\mathbf{p}_m = j\eta_o(\mathbf{p}_+ - \mathbf{p}_-), \qquad (34)$$

the electric and magnetic dipole moments are

$$\begin{pmatrix}\mathbf{p}_e\\ \mathbf{p}_m\end{pmatrix} = \begin{bmatrix}\alpha_{ee} & \alpha_{em}\\ \alpha_{me} & \alpha_{mm}\end{bmatrix}\begin{pmatrix}\mathbf{E}_o\\ \mathbf{H}_o\end{pmatrix} \qquad (35)$$

where

$$\alpha_{ee} =$$
$$\frac{4\pi\epsilon_0 a_1^3}{\Delta}\Big\{\Big(\big[(\mu_{r1}+2)(\epsilon_{r1}-1)-\kappa_1^2\big]\delta+\big[(\mu_{r1}-1)(2\epsilon_{r1}+1)-2\kappa_1^2\big]\Big)$$
$$\cdot\Big(\big[(\mu_{r2}+2\mu_{r1})(\epsilon_{r2}+2\epsilon_{r1})-(\kappa_2+2\kappa_1)^2\big]\delta$$
$$+\,2\big[(\mu_{r2}-\mu_{r1})(\epsilon_{r2}-\epsilon_{r1})-(\kappa_2-\kappa_1)^2\big]\Big)$$
$$-\,9\delta\big[2(\epsilon_{r1}\mu_{r1}-\kappa_1^2)^2+(\epsilon_{r1}\mu_{r1}-\kappa_1^2)(\mu_{r2}-2\epsilon_{r2})+\kappa_2^2-\epsilon_{r2}\mu_{r2}\big]\Big\} \tag{36}$$

$$-\alpha_{em}=\alpha_{me}=\frac{12\pi j\sqrt{\mu_0\epsilon_0}\,a_1^3}{\Delta}\Big\{(\kappa_1\delta+2\kappa_1)$$
$$\cdot\Big(\big[(\mu_{r2}+2\mu_{r1})(\epsilon_{r2}+2\epsilon_{r1})-(\kappa_2+2\kappa_1)^2\big]\delta$$
$$-\,\big[(\mu_{r2}-\mu_{r1})(\epsilon_{r2}-\epsilon_{r1})-(\kappa_2-\kappa_1)^2\big]\Big)$$
$$+\,9\delta\big[\kappa_1^2(\kappa_1+\kappa_2)+\epsilon_{r1}\mu_{r1}(\kappa_2-\kappa_1)-\kappa_1(\epsilon_{r1}\mu_{r2}+\epsilon_{r2}\mu_{r1})\big]\Big\} \tag{37}$$

$$\alpha_{mm} =$$
$$\frac{4\pi\mu_0 a_1^3}{\Delta}\Big\{\Big(\big[(\mu_{r1}-1)(\epsilon_{r1}+2)-\kappa_1^2\big]\delta+\big[(2\mu_{r1}+1)(\epsilon_{r1}-1)-2\kappa_1^2\big]\Big)$$
$$\cdot\Big(\big[(\mu_{r2}+2\mu_{r1})(\epsilon_{r2}+2\epsilon_{r1})-(\kappa_2+2\kappa_1)^2\big]\delta$$
$$+\,2\big[(\mu_{r2}-\mu_{r1})(\epsilon_{r2}-\epsilon_{r1})-(\kappa_2-\kappa_1)^2\big]\Big)$$
$$-\,9\delta\big[2(\epsilon_{r1}\mu_{r1}-\kappa_1^2)^2+(\epsilon_{r1}\mu_{r1}-\kappa_1^2)(\epsilon_{r2}-2\mu_{r2})+\kappa_2^2-\epsilon_{r2}\mu_{r2}\big]\Big\} \tag{38}$$

and Δ is defined in equation (29).

3. Discussion of the Result

First of all, the results must be checked with previously known special cases. The polarizability matrix of an homogeneous chiral sphere is known [4,8]. There are several ways to reduce the present two-layer case to an homogeneous chiral sphere:

- $\kappa_1 \to 0, \epsilon_{r1} \to 1, \mu_{r1} \to 1$: only the core is chiral.
- $\delta \to 1$: the layer thickness vanishes.
- $\delta \to \infty$: the inner sphere radius vanishes, only one sphere consisting of medium 1 remains.
- $\kappa_1 \to \kappa_2, \epsilon_{r1} \to \epsilon_{r2}, \mu_{r1} \to \mu_{r2}$: the shell and the core are of the same chiral material.

It is straightforward, albeit tedious, to show that indeed in all these four cases the polarizability matrix reduces to

$$4\pi a^3 \begin{bmatrix} \epsilon_0 \dfrac{(\mu_r+2)(\epsilon_r-1)-\kappa^2}{(\mu_r+2)(\epsilon_r+2)-\kappa^2} & \dfrac{-3j\kappa\sqrt{\mu_0\epsilon_0}}{(\mu_r+2)(\epsilon_r+2)-\kappa^2} \\ \dfrac{3j\kappa\sqrt{\mu_0\epsilon_0}}{(\mu_r+2)(\epsilon_r+2)-\kappa^2} & \mu_0 \dfrac{(\mu_r-1)(\epsilon_r+2)-\kappa^2}{(\mu_r+2)(\epsilon_r+2)-\kappa^2} \end{bmatrix} \tag{39}$$

which is the same result as in [4]. Note that $\kappa, \epsilon_r, \mu_r$ can be the parameters of core or shell according to the case.

Another special case is the layered dielectric (or magnetic) sphere: $\kappa_1 \to 0, \kappa_2 \to 0$. In this case the polarizability matrix becomes diagonal: the magnetoelectric terms vanish. The coupling disappears: the electric polarizability depends only on the permittivities, while the magnetic polarizability depends only on the permeabilities:

$$4\pi a_1^3 \begin{bmatrix} \epsilon_0 \dfrac{(1+2\epsilon_{r1})(\epsilon_{r2}-\epsilon_{r1})+\delta(\epsilon_{r1}-1)(2\epsilon_{r1}+\epsilon_{r2})}{2(\epsilon_{r1}-1)(\epsilon_{r2}-\epsilon_{r1})+\delta(2+\epsilon_{r1})(2\epsilon_{r1}+\epsilon_{r2})} & 0 \\ 0 & \mu_0 \dfrac{(1+2\mu_{r1})(\mu_{r2}-\mu_{r1})+\delta(\mu_{r1}-1)(2\mu_{r1}+\mu_{r2})}{2(\mu_{r1}-1)(\mu_{r2}-\mu_{r1})+\delta(2+\mu_{r1})(2\mu_{r1}+\mu_{r2})} \end{bmatrix} \tag{40}$$

which is the same result in [9].

A third special case, although not studied earlier, is the situation where the determinant Δ in equation (20) reduces to a product of two terms. This happens if $\kappa_1^2 \to \epsilon_{r1}\mu_{r1}, \kappa_2^2 \to \epsilon_{r2}\mu_{r2}$. This is a rather extreme case of strong chirality, where the chirality parameter equals the refractive index in both media. In this case, the polarizabilities reduce to the following form:

$$2\pi a_1^3 \begin{bmatrix} \epsilon_0 \dfrac{-1-2\epsilon_{r1}+\mu_{r1}+\delta(-2+2\epsilon_{r1}-\mu_{r1})}{1-\epsilon_{r1}-\mu_{r1}+\delta(2+\epsilon_{r1}+\mu_{r1})} & \dfrac{3j\kappa_1(1-\delta)\sqrt{\mu_0\epsilon_0}}{1-\epsilon_{r1}-\mu_{r1}+\delta(2+\epsilon_{r1}+\mu_{r1})} \\ \dfrac{-3j\kappa_1(1-\delta)\sqrt{\mu_0\epsilon_0}}{1-\epsilon_{r1}-\mu_{r1}+\delta(2+\epsilon_{r1}+\mu_{r1})} & \mu_0 \dfrac{-1+\epsilon_{r1}-2\mu_{r1}+\delta(-2-\epsilon_{r1}+2\mu_{r1})}{1-\epsilon_{r1}-\mu_{r1}+\delta(2+\epsilon_{r1}+\mu_{r1})} \end{bmatrix} \tag{41}$$

It is also worth noting the reciprocity of the polarizability matrix: the crosspolarizability terms are complex conjugates of each other for lossless case. Chiral media are reciprocal [10]. Also the effect of the handedness (the sign of the chirality parameter) on the polarizability components is logical: changing the sign of κ_1 and κ_2 changes the sign of the crosspolarizabilities α_{em} and α_{me}, and it will also change the sign of the macroscopic chirality of a medium containing this type of layered chiral spheres. Furthermore, it is in accordance with intuition that the sign change in κ_1 and κ_2 does not affect the magnitude of the polarizabilities α_{ee} and α_{mm}.

As a numerical example, finally the shielding effect of a dielectric layer is calculated. Given a chiral sphere with known polarizability matrix, the expressions derived in this article are used to examine how the polarizability components change as the layer permittivity changes. The results are illustrated in Figs. 2a, 2b, and 2c. To see the main effects, the permeabilities of both media are assumed to be those of free space $(\mu_1 = \mu_2 = \mu_0)$. The chirality parameter of the core is rather small: $\kappa_2 = 0.1$.

Figure 2a shows the magnitude of the electric polarizability α_{ee}, and it is clear that with increasing permittivity — be it in the layer or

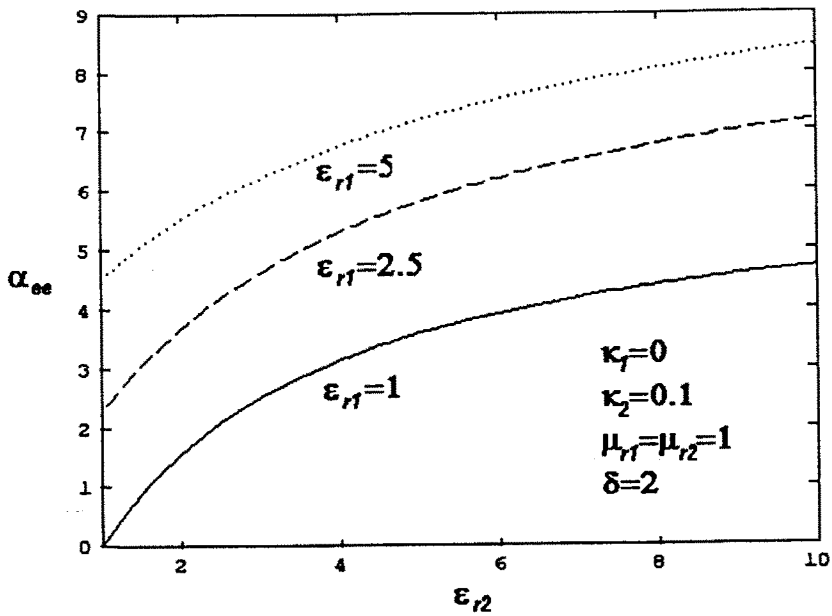

Figure 2a. Shielding effect of a dielectric layer on the electric polarizability α_{ee} of a chiral sphere. The value shown is $\alpha_{ee}/\epsilon_0 a_1^3$. The permeabilities of the core and the shell are assumed to be those of free space and the cube of the ratio of the radii is $\delta = a_1^3/a_2^3 = 2$. The chirality parameter of the core is $\kappa_2 = 0.1$.

in the core — the polarizability increases. Figure 2b shows an interesting effect: the appearance of magnetic polarization in a chiral sphere without magnetic susceptibility. It is true that the magnetic polarizability is small but it is finite. This counterintuitive fact has also been noted earlier and underlined recently in the study of homogeneous chiral spheres [11]. Note the sign of α_{mm} : it is negative, in other words this coated chiral sphere displays a diamagnetic character.

Finally, Fig. 2c shows the effect of layer permittivity on the crosspolarizability α_{em} . The message is a rather strong shielding effect: α_{em} decreases strongly as ϵ_1 increases from the vacuum-permittivity value (no layer). On the other hand, the figure also witnesses that α_{em} is decreased with increasing core permittivity, a fact that has been earlier discussed in [12].

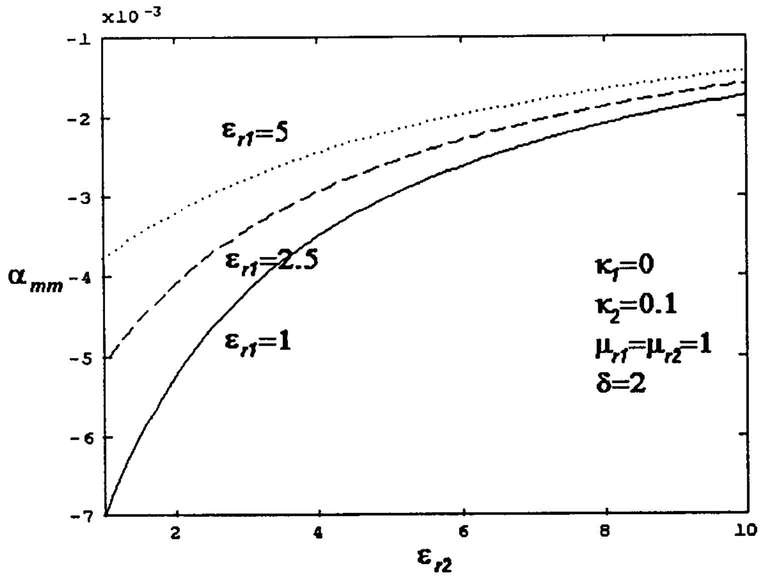

Figure 2b. The same as Figure 2a, for the magnetic polarizability. The value shown is $\alpha_{mm}/\mu_0 a_1^3$.

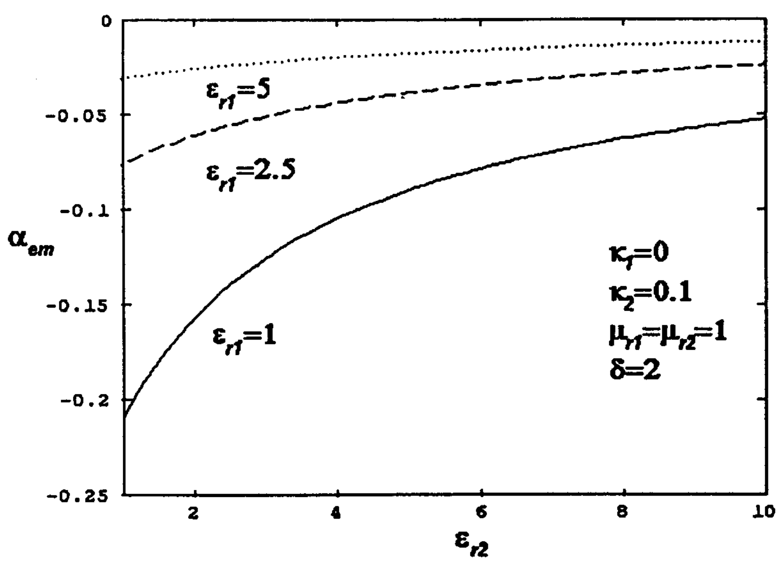

Figure 2c. The same as Figure 2a, for the crosspolarizability. The value shown is $\alpha_{em}/j\sqrt{\mu_0\epsilon_0}a_1^3$.

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