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ELECTROMAGNETIC THEORY BASED ON INTEGRAL REPRESENTATION OF FIELDS AND ANALYSIS OF SCATTERING BY OPEN BOUNDARY

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1. Introduction

Scattering of electromagnetic wave by an obstacle has usually been analyzed theoretically by a boundary value problem of the Helmholtz equation or of the Maxwell equations. In most cases, a boundary, or a surface of an obstacle, was a closed one such as a circle in the two-dimensional case or a sphere in the three-dimensional case. On the other hand, there are many important problems where a boundary is open such as a semi-circle or a hemi-sphere. Reflectors of antennas, a hollow pipe with slots, some kind of grating, are other examples of an open boundary. Though many works have been done on various closed boundary problems, fewer have been done on the theoretical study of open boundary problems.

There are two purposes of the present paper. The first one is to give representation formulas to electromagnetic fields in a domain bounded by a closed as well as an open boundary, in a necessary and sufficient form to represent the fields, and, on the basis of the formulas, to reconsider the traditional concepts such as incident fields, radiation conditions and edge conditions, obtaining generalized radiation conditions and generalized edge conditions. Eigen fields in an unbounded domain is also discussed.

The second one is to construct, with the help of the representation formulas, a general theory of electromagnetic fields in a domain bounded by an open boundary, and on the basis of the mathematical background thus obtained, to give a general practical procedure to analyze the fields numerically.

Problems on a closed boundary are also studied, because they are helpful to make the distinction of an open boundary problems clear.

The characteristic feature of a closed boundary is that it separates the whole space into two domains — the interior and the exterior. As soon as a part of a closed boundary is removed, its interior domain disappears, and sharp edges newly appear there, and make an open boundary. The disappearance of the interior and appearance of edges make the study of electromagnetic scattering by an open boundary difficult in comparison with the one by a closed boundary.

There are two kinds of approach to a boundary value problem. One is by an integral equation, and the other is by a series expansion. A comparison between the procedures for closed and open boundaries with respect to these approaches will help to show how an open boundary problem is difficult.

The Dirichlet problem of the two-dimensional Helmholtz equation is a mathematical model of scattering of E wave. Weyl [1] and Müller [2] assumed a solution of the Dirichlet exterior problem of the Helmholtz equation for a closed boundary to be a "double layer potential" with an unknown density distributed over the boundary, and reduced the problem to that of solving for an integral equation of Fredholm of the second kind with respect to the density, and thus established a general theory for a closed boundary. The merit of their method was that they could reduce the problem to that of an integral equation of the second kind whose analysis has been well studied. However, as will be proved later in Section 6.10, their method by a double layer potential was available because of the existence of an interior domain, and hence, it is not effective for the case of an open boundary. Instead, a solution for an open boundary is necessarily represented by a "simple layer potential" and the problem is necessarily reduced to the one solving for an integral equation of the first kind, which is an ill-posed and misleading problem and is not so well studied. Furthermore, due to the singularity at edge points of an open boundary, we must be careful in making use of the L_2 theory such as a series expansion in the study of an integral equation of the first kind.

Contrary to Weyl's integral equation approach, Calderón [3] and Yasuura [4], independently developed a series expansion method to solve an exterior problem for a closed boundary. They considered an aggregate of Green's functions and their derivatives, whose singularities, or source points, being located in the interior of a closed boundary. Then, they proved that an appropriate combination of them gives an approximate solution which converges to the true solution in the exterior of the closed boundary. However, their method is also of no use in an open boundary problem because there is no interior domain where the singularities of their functions are to be placed. Consequently, if a series expansion method is expected to solve an open boundary problem, it is necessary to construct a new set of functions which have no singularity at all in the outside of the open boundary, and moreover, play the same role as Calderón and Yasuura's functions.

The author has developed his integral equation approach for an open boundary which corresponds to the work by Weyl and Müller for a closed boundary, and also his series expansion approach for an open boundary which corresponds to the work by Calderón and Yasuura for a closed boundary.

In 1964, in connection with a communication trouble which a spacecraft suffers at its reentry to the atmosphere, the author solved some boundary value problems for the two dimensional Helmholtz equation for an open boundary composed of coaxial circular arcs. He reduced the problems to the one that solves an integral equation of Fredholm of the first kind, which he converted to a singular integral equation with Cauchy kernel and solved it [5,6]. At that time, he and Lewin [7], independently, were the first who introduced a singular integral equation technique to the study of electromagnetic field. Later, he improved his method so that it is applicable to general geometry of an open boundary [8]. He worked on a series expansion approach as well [9,10]. He introduced a set of certain functions which he called "elemental electromagnetic field functions." With help of these functions, he established his expansion approach to an open boundary problem, which corresponds to the approach by Calderón and Yasuua for a closed boundary problem. The detail of which will be described in Section 5 below.

Integral representation formulas which will be introduced in Sections 3 and 4 are a mathematical version of Huygens' principle. In Section 2, traditional concepts such as incident fields, radiation conditions and edge conditions, will be reviewed on the basis of representation formulas, that is, from the view points of Huygens' principle. Thanks to the representation formulas, we shall see that primary and secondary fields are separated naturally and smoothly. Also, we shall have a deeper understanding on radiation conditions and edge conditions.

Sommerfeld's radiation condition was introduced to make a solution of a field equation unique, discriminating outgoing and incoming waves, and thus examining if no source exists at infinity. Edge condition was another condition for the uniqueness of a solution, depending on a physical assumption of finite energy in a neighborhood of an edge.

The author will assert that sources may exist at infinity and infinite energy at an edge may be permissible, and then, with the help of the representation formula, he will introduce generalize radiation conditions and generalized edge conditions, which still make the corresponding solution unique.

As was shown above, an integral representation formulation of a field is indispensable in a study of a boundary value problem, especially when a boundary is of irregular geometry such as an open boundary. If one's representation formula is only of necessary form to be a field, he may suffer from a spurious solution. However, our integral representation formulas will be of necessary and sufficient form in the sense that any field must be represented by the formulas and conversely, a function defined by the formulas satisfies all requirements prescribed for the function to be a field. Thanks to the necessary and sufficient property of our formulas, we are free from a spurious solution.

Continuity relations between surface electric current and charge densities, which are the necessary and sufficient conditions for the representation formulas to be electromagnetic fields, will be studied. In the case of an open boundary, a simple condition which requires surface electric current to be tangential to the periphery of an open boundary will be introduced, and will be shown to be equivalent to the continuity condition between surface electric current and charge densities. It is noted that our condition is the one which is required to hold on a line, while the traditional continuity condition is required on a whole surface. This makes our condition advantageous in numerical calculations.

In Section 5, the authors theory of a series expansion approach to solve electromagnetic fields in a domain bounded by an open boundary, which is another main theme of this paper, will be developed. Two and three dimensional problems will be treated. "Elemental electromagnetic field function" which play the most important role in the approach will be introduced. It will be proved that a pertinent linear combination of them can be chosen so that it represents electromagnetic fields in a domain outside of an open boundary, and that it approximates the boundary value of arbitrarily given incident fields on the boundary as precisely as wanted, thus showing that the linear combination furnishes us with approximate fields which converge to the true fields corresponding to the given incident fields uniformly in the domain outside of the open boundary. On the basis of this result, a practical procedure of numerical calculation, which is applicable to any incident fields and to any geometry of an open boundary, will be given.

In Section 6, detailed proof of mathematical results, which have appeared in the preceding sections without proof, will be given. However, Section 6 will be devoted not only to such proofs, but also to the presentation of some important mathematical results. In Section 6.10, various integral equations relating to closed and open boundary problems will be introduced. The well known method of Weyl, which reduces a boundary value problem to that of solving for an integral equation of Fredholm of the second kind, will be discussed, and the method will be proved not to be applicable to an open boundary problem. In Section 6.11, various theorems on and related to the unique determination of fields will be proved. They will play an important role in our expansion approach. In Section 6.14, continuity relations between surface electric current and charge densities, and relating important theorems as well, will be studied. In Section 6.18, supplemental explanation on some theorems and mathematical terminology will be given. Also, the authors expansion method will be discussed in relation to other method.

2. Field Equations and Related Subjects

2.1 The Maxwell Equations.

We postulate that any and every electromagnetic phenomenon is governed by the Maxwell equations which are

$$\begin{aligned} &\nabla \times \tilde{E} = -\partial \tilde{B}/\partial t, \quad \nabla \times \tilde{H} = \tilde{J} + \partial \tilde{D}/\partial t, \\ &\nabla \cdot \tilde{B} = 0, \quad \nabla \cdot \tilde{D} = \rho \end{aligned} \tag{1}$$

where $\tilde{E}$, $\tilde{B}$, $\tilde{H}$, $\tilde{D}$, and $\tilde{J}$ are three-dimensional vector functions of space and time called electric field strength, magnetic flux density, magnetic field strength, electric displacement and space electric current density, respectively, while ρ is a scalar function called space electric charge density. We generically call them field functions or simply fields. As is well known, a set of these equations forms a mathematical model of Faraday's induction law and Ampere's circuit law supplemented with displacement current by Maxwell. We assume that all of field functions appearing in the equations are bounded and have continuous second order derivatives at points where they are considered.

These postulations have been guaranteed to be true by the fact that any experimental evidence which contradicts them has never existed yet.

2.2 Constitutive Relations, Time Harmonic Fields.

Since there are only two vector equations and two scalar equations governing five vector functions and one scalar function, we need some more additional conditions imposed on them so that the system of these equations and conditions is to be made determinate. Furthermore, any information concerning the properties of a medium where an electromagnetic phenomenon is considered is not furnished by equations (1). Thus, we must employ some functional relations between the field functions characterizing the physical properties of the medium, which we call constitutive relations. There are various constitutive relations corresponding to the variety of media, which will be found in such text books as Stratton [11] and Kong [12]. In the present work, we are concerned only with the so called homogeneous and isotropic medium, where the constitutive relations are

$$\tilde{D} = \epsilon \tilde{E}, \quad \tilde{B} = \mu \tilde{H}, \quad \tilde{J} = \sigma \tilde{E} \tag{2}$$

where ϵ, μ and σ are constants named permittivity, permeability and conductivity, respectively. These relations imply that the medium is of uniform character at any point and in any direction in it.

Substituting (2) for the equations (1), we have

$$\begin{aligned} &\nabla \times \tilde{E} = -\mu \partial \tilde{H}/\partial t, && \nabla \times \tilde{H} = \sigma \tilde{E} + \epsilon \partial \tilde{E}/\partial t, \\ &\nabla \cdot \tilde{H} = 0, && \nabla \cdot \tilde{E} = \rho/\epsilon \end{aligned} \tag{3}$$

Assume that field vectors are periodic functions of time with period $T = 2\pi/\omega$, and are expanded in a Fourier series as $\tilde{E} = \sum \tilde{E}_n e^{in\omega t}$ and $\tilde{H} = \sum \tilde{H}_n e^{in\omega t}$. Then, the first equation of (3) becomes $\sum \nabla \times \tilde{E}_n e^{in\omega t} = -\mu \sum in\omega \tilde{H}_n e^{in\omega t}$, which is shown to be equivalent, by virtue of the orthogonality property of $\{e^{in\omega t}\}$, to $\nabla \times \tilde{E}_n = -in\omega\mu$ $\tilde{H}_n$. In a way similar to this, the second equation of (3) is reduced to $\nabla \times \tilde{H}_n = (\sigma + in\omega\epsilon)\tilde{E}_n$.

If $\tilde{E}_n$, $\tilde{H}_n$, $n\omega$ and $(\sigma + in\omega\epsilon)$ are replaced by $\tilde{E}$, $\tilde{H}$, ω and $i\omega\epsilon$, respectively, then (3) is rewritten as

$$\begin{aligned} \nabla \times \tilde{E} &= -i\omega\mu\tilde{H}, & \nabla \times \tilde{H} &= i\omega\epsilon\tilde{E}, \\ \nabla \cdot \tilde{E} &= 0, & \nabla \cdot \tilde{H} &= 0 \end{aligned} \tag{4}$$

Conversely, if a solution of (4) is obtained, and if ω and $i\omega\epsilon$ are replaced by $n\omega$ and $(\sigma + in\omega\epsilon)$, respectively, then it gives the n-th term of the Fourier expansion of solutions of the original equation (3). Hence, (4) is the Maxwell equations for time harmonic fields in a homogeneous and isotropic medium.

It is noted that the last two equations of (4) hold automatically if the first two equations of (4) hold.

2.3 Two-Dimensional Field, The Helmholtz Equation.

From the equations (4), it follows that $\nabla \times \nabla \times \tilde{E} = -\nabla \cdot \nabla \tilde{E} = \omega^2 \epsilon\mu \tilde{E}$. That is, an electric field vector $\tilde{E}$ satisfies a vector Helmholtz equation

$$\triangle \tilde{E} + k^2 \tilde{E} = 0 \tag{5}$$

where we have set

$$k^2 = \omega^2 \epsilon\mu \qquad \mathrm{Im}\, k \le 0 \tag{6}$$

Conversely, if a solution $\tilde{E}$ of (5) is to be an electric field, it is natural to define a magnetic field $\tilde{H}$ by $\tilde{H} = (i/\omega\mu)\nabla \times \tilde{E}$. However, $\tilde{E}$ and $\tilde{H}$ thus defined do not necessarily satisfy (4), while the necessary and sufficient condition that they satisfy (4) is the third equation of (4), namely, $(4)_3; \nabla \cdot \tilde{E} = 0$. That is, the pair of the first two equations of (4) is equivalent to the pair of $(4)_3$ and (5). In other words, electromagnetic fields $\tilde{E}$ and $\tilde{H}$ are obtained by a solution $\tilde{E}$ of (5) and $(4)_3$.

We have a simple solution of these equations (5) and $(4)_3$ as follows. Let (x, y, z) be a rectangular coordinate system, and $\tilde{i}_z$ be the unit vector along z-axis. Assume that u is a function of x and y only, which satisfies the two-dimensional Helmholtz equation

$$\triangle u + k^2 u = 0 \tag{7}$$

and set $\tilde{E} = \tilde{i}_z u$. Then, it is easy to see that $\triangle \tilde{E} + k^2 \tilde{E} = \tilde{i}_z(\triangle u + k^2 u) = 0$ and that $\nabla \cdot \tilde{E} = \tilde{i}_z \cdot \nabla u = (\partial u/\partial z) = 0$. That is, $\tilde{E}$ thus defined satisfies both (5) and $(4)_3$. Consequently, $\tilde{E} = \tilde{i}_z u$ and $\tilde{H} = (-i/\omega\mu)\tilde{i}_z \times \nabla u$ satisfy the equation (4) and make electromagnetic fields. Obviously, $\tilde{E}_t = \tilde{E} - \tilde{i}_z(\tilde{i}_z \cdot \tilde{E}) = 0$ and $H_z = \tilde{i}_z \cdot \tilde{H} = 0$. We call them a E wave. Thus, the two-dimensional Helmholtz equation (7) gives a mathematical model of a E wave.

2.4 Boundary Conditions and Boundary Value Problems.

Though field vectors have been assumed, as was mentioned above, to be continuous everywhere in a domain where a medium varies continuously, discontinuity of fields may occur on a surface at which the physical properties of a medium change abruptly.

Let S be a surface which separates two media 1 and 2, and $\tilde{n}$ be the unit vector normal to S. Then, as is found in ordinary text books, the following relations, which determine the transition of field vectors from one medium to another, are required to hold at a point on a boundary S.

$$\begin{aligned} &\tilde{n} \times (\tilde{E}_1 - \tilde{E}_2) = 0, \qquad \tilde{n} \times (\tilde{H}_1 - \tilde{H}_2) = \tilde{K}, \\ &\tilde{n} \times (\tilde{B}_1 - \tilde{B}_2) = 0, \qquad \tilde{n} \cdot (\tilde{D}_1 - \tilde{D}_2) = \varpi \end{aligned} \tag{8}$$

In (8), a quantity with suffix $j\ (=1,2)$ is a field vector in the medium j. While, $\tilde{K}$ is a surface electric current density, and ϖ is a surface electric charge density.

If the conductivity σ in a medium is infinite, no fields are able to exist. The proof is as follows; if $\sigma = \infty$, $i\omega\epsilon$ which is originally $(\sigma + i\omega\epsilon)$, becomes infinity. While $\nabla \times \tilde{H}$ has been assumed to be finite. Hence from the second equation of (4), we have $\tilde{E} = 0$. If $\tilde{E}$ is zero everywhere in a vicinity of a point, its derivatives are also zero at that point, which implies that $\tilde{H} = (i/\omega\mu)\nabla \times \tilde{E} = 0$ there. A medium whose conductivity is infinite is called a perfect conductor. If S is a surface of a perfect conductor, conditions (8) are reduced to

$$\tilde{n} \times \tilde{E} = 0, \quad \tilde{n} \cdot \tilde{H} = 0, \quad \tilde{n} \times \tilde{H} = \tilde{K}, \quad \tilde{n} \cdot \tilde{E} = \varpi/\epsilon \tag{9}$$

So, a mathematical model of electromagnetic fields in a domain bounded by a surface S is given by the boundary value problem of the Maxwell equations (4) accompanied by the boundary conditions (8) or (9).

On the other hand, let S be a rectilinear surface parallel to a z-axis, whose intersection with a plane perpendicular to the z-axis being C, and sources which generate a field $\tilde{E}$ be distributed uniformly along the z-axis. Then, since the geometry of a boundary, and of sources as well, are independent of z, $\tilde{E}$ is also independent of z, and a two-dimensional wave $\tilde{E} = \tilde{i}_z u$ exists.

Boundary condition on C for u is obtained from (8) and (9) as

$$u_1 = u_2 \tag{10}$$

when C separates two non perfectly conducting media 1 and 2, and

$$u = 0 \tag*{(10)$'$}$$

when C is a boundary of a perfect conductor.

So, a mathematical model of a two-dimensional E wave is the boundary value problem of the Helmholtz equation (7) accompanied by the condition (10) or (10)$'$.

In order that the boundary value problems mentioned above determine electromagnetic fields definitely, additional conditions such as radiation condition and edge condition may in some cases be necessary. On the other hand, we have to consider how sources which give rise to the fields should be treated. These will be discussed in the following, with the help of the fundamental integral representation formulas introduced below.

2.5 Fundamental Representation Formulas of Fields.

The author's theory developed in this work is based fundamentally on the integral representation formulas of fields. Though the precise formulas will be given later in Sections 3 and 4, an outline of them will be described here.

Let S and S^* be closed surfaces such that each of which exists in the outside of the other. Assume that sources which generate electromagnetic fields are enclosed by S^*, and that all of S and S^* are enclosed by a surface S_0. Let the medium enclosed by S, S^* and S_0 be homogeneous and isotropic. Then, by the integral representation formulas (33)$'$ in Section 3.2, electromagnetic fields $\tilde{E}^t$ and $\tilde{H}^t$ in the bounded medium surrounded by S, S_0 and S^*, where superscript t implies total fields, are represented by certain integrals taken over these surfaces. That is, denoting both of $\tilde{E}^t$ and $\tilde{H}^t$ generically as $\tilde{F}^t$, we have

$$\tilde{F}^t = \tilde{I}(S + S_0) + \tilde{I}(S^*),$$

where $\tilde{I}$ is a certain integral taken over the surface indicated in parenthesis. For details, see (38) and the notes following it.

This is the fundamental representation formula for electromagnetic fields $\tilde{F}^t$, that is, for $\tilde{E}^t$ and $\tilde{H}^t$, in a bounded medium.

If a medium is exterior to S and S^*, and hence is unbounded, we enclose S and S^* by a sphere $S(R)$ of sufficiently large radius R. Then, it is obvious from what was stated above, that total fields in the medium are represented by

$$\tilde{F}^t = \tilde{I}(S) + \tilde{I}(S^*) + \tilde{I}(S(R)).$$

For details, see (39) and the notes following it.

This is the fundamental representation formula for electromagnetic fields $\tilde{F}^t$, or, for $\tilde{E}^t$ and $\tilde{H}^t$, in an unbounded medium.

In these formulas, $\tilde{I}(S^*)$ and $\tilde{I}(S(R))$ are, respectively, the contributions to total fields made by sources or singularities located in the inside of S^* and outside of $S(R)$, and $\tilde{I}(S(R))$ will be the subject of discussion on radiation conditions in Section 2.7. For details, see Sections 2.7 and 2.9.

These results stated above for the case where S is a closed surface are applied to obtain similar representation formulas for the case where S is an unclosed surface.

If S is unclosed like a disk of zero thickness, S has sharp edges such as a periphery of a disk. We enclose the edges by curved circular-cylindrical surfaces $S(\rho)$ of a small radius ρ, whose central lines being the lines of the edges. Then, as will be shown in Section 4, fields $\tilde{F}^t$ at a point in the medium outside of S, S^* and $S(\rho)$ are proved to be represented by

$$\tilde{F}^t = \tilde{I}(S) + \tilde{I}(S^*) + \tilde{I}(S(\rho)) + \tilde{I}(S(R)),$$

where $\tilde{I}$ denotes certain integrals taken over a surface indicated in parenthesis.

This is the fundamental representation theorem for the case of an unclosed boundary, which will give the basis of the analysis of scattering by an open boundary developed in Section 5 below.

$\tilde{I}(S(\rho))$ will be the subject of discussion on edge conditions in Section 2.8 below. See also Section 2.9.

Note that all of these representation formulas mentioned above are of necessary and sufficient form to represent the fields in the sense that any solutions of the Maxwell equations are necessarily represented by the formulas, and conversely, the functions defined by the formulas satisfy the Maxwell equations. In other words, the fields are equivalent to the functions represented by these formulas and there are no other fields.

From the point of view of physics, the integral representation formulas introduced above are a mathematical version of Huygens' principle. That is, $\tilde{I}(S^*)$, $\tilde{I}(S(\rho))$ and $\tilde{I}(S(R))$ represent effects of sources or singularities located in the inside of S^* and $S(\rho)$, and outside of $S(R)$, respectively, in terms of current and charge densities induced on these surfaces.

Similar results are obtained for the case of a two-dimensional E wave. Suppose that C and C^* are closed curves in a plane such that each of which exists in the outside of the other, that sources of fields are enclosed in the inside of C^*, and that C and C^* are enclosed by a closed curve C_0. Let a medium encircled by C, C^* and C_0 be homogeneous and isotropic. Then, a E wave u^t in the closed medium between C, C^* and C_0, where superscript t implies a total field, is represented by the formula $(18)'$ of Section 3.1 below, which is of the form

$$u^t = I(C + C_0) + I(C^*),$$

where I denotes a certain integral taken over a curve indicated in the parenthesis.

This is the fundamental integral representation formula for a E wave in a bounded medium.

If a medium is exterior to C and C^*, and hence is unbounded, we enclose C and C^* by a circle $C(R)$ of a sufficiently large radius R. Then, as will be shown by $(19)'$ in Section 3.1, u^t at a point in the medium surrounded by C, C^* and $C(R)$ is represented by

$$u^t = I(C) + I(C^*) + I(C(R)).$$

This is the fundamental integral representation formula for a E wave in an unbounded medium.

In these formulas, $I(C^*)$ and $I(C(R))$ represent contributions to total fields by sources or singularities located in the inside of C^*

and outside of $C(R)$, respectively. $I(C(R))$ will be the subject of discussion on radiation conditions in Section 2.7 below.

With the help of these results for the case of a closed curve C, representation formulas are obtained for the case where C is unclosed line like a semi-circle etc. Since an unclosed line C has end points, we enclose each one of them by a circle of small radius ρ, and denote the aggregate of them as $C(\rho)$. Then, as will be shown in Section 4, a E wave u^t in a medium in the exterior of C, C^* and $C(\rho)$ is represented by the formula,

$$u^t = I(C) + I(C^*) + I(C(R)) + I(C(\rho))$$

where $I(\ )$'s are certain integrals on a curve indicated in the parenthesis, though not the same as those appeared in formulas for a closed C. $I(C(\rho))$, which represents the effect of singularity at end points, will be the subject of discussions in Section 2.8 on edge conditions.

This formula will be the basis of the analysis of scattering of a E wave by an open boundary C, which will be developed in Section 5.

2.6 Primary Incident Fields and Secondary Scattered Fields.

One way, which is frequently taken to express primary fields or source terms, is to assume $\tilde{J}$ and ρ appearing in the equations (4) to be known functions given by sources.

However, we shall not take this way. Instead, we shall separate total fields into the sum of primary or incident fields and secondary or scattered fields. Primary fields are assumed to be known functions given exclusively by sources, which satisfy the Maxwell equations everywhere in the whole space except at given source points where the primary fields are generated. On the other hand, secondary fields are considered to be differentiable functions which satisfy the Maxwell equations everywhere in the medium even at the site of sources where primary fields may not be finite, and to be subjected to the prescribed boundary conditions at the boundary of the medium. Such a separation of total fields into primary and secondary fields is realized smoothly by the integral representation formulas obtained in the preceding section.

For instance, as was mentioned in the preceding section, total fields $\tilde{F}^t$ in an unbounded medium exterior to a closed surface S is represented by the sum of $\tilde{I}(S^*) + \tilde{I}(S(R))$ and $\tilde{I}(S)$, where $\tilde{I}(S^*)$ and $\tilde{I}(S(R))$ are the contributions to $\tilde{F}^t$ by sources located in the inside of S^* and in the outside of $S(R)$, respectively, and as will be discussed in Section 2.9 below, are assumed to be known functions given in terms of sources. We consider the sum of them to be the primary fields because they are conformed to the requirements to be primary fields stated above. While, $\tilde{I}(S)$ is taken to be the secondary fields. Similarly, in an unbounded medium in the outside of an unclosed surface S, primary fields are assumed to be given by $\tilde{I}(S^*)+\tilde{I}(S(R))+\tilde{I}(S(\rho))$, where, as will be discussed in Section 2.9, $\tilde{I}(S(\rho))$ is a known function representing the effect of singularity induced on edges of S, while secondary fields are assumed to be given by $\tilde{I}(S)$.

In the case of a two-dimensional E wave in a medium exterior to a closed curve C, a primary field is defined by $I(C^*)+I(C(R))$, while a secondary filed is assumed to be $I(C)$. On the other hand, if C is unclosed, a primary field is assumed to be $I(C^*)+I(C(R))+I(C(\rho))$, and a secondary field is assumed to be $I(C)$.

These results obtained above will play important roles in the discussions on the analysis of a mathematical model of fields in Section 2.9 below.

2.7 Traditional Radiation Conditions.

In order to determine fields in a unbounded medium definitely and uniquely, some conditions must be imposed on the behavior of fields at infinity.

As such conditions, Sommerfeld's radiation condition and condition of finiteness at infinity have usually been employed in a problem for the Helmholtz equation (7), which are

$$\partial u^t(P)/\partial R + iku^t(P) = 0(R^{-1}), \quad u^t(P) = 0(R^{-1}) \tag{11}$$

where $u^t(P)$ is the value of u^t evaluated at a point P, and R is the distance of P from an arbitrarily fixed origin O. On the other hand, in a three dimensional vector boundary value problem for the Maxwell equations (4), a vector version of (11) such as

$$\begin{aligned} [\tilde{r} \times \tilde{E}^t - (k/\omega\epsilon)\tilde{H}^t] &= \tilde{0}(R^{-1}), \quad \tilde{E}^t = \tilde{0}(R^{-1}), \\ [\tilde{r} \times \tilde{H}^t + (k/\omega\mu)\tilde{E}^t] &= \tilde{0}(R^{-1}), \quad \tilde{H}^t = \tilde{0}(R^{-1}), \end{aligned} \tag{12}$$

where $\tilde{r} = \overrightarrow{\mathrm{OP}}/\,|\mathrm{OP}|$, have been employed by Müller [13], Kong [12], etc.

We do not adopt these conditions, but employ the conditions due to Wilcox [14] such as

$$\int_{C(R)} |\partial u^t(P)/\partial R + iku^t(P)|^2 ds_P = 0(R^{-1}) \tag{13}$$

instead of (11), and those by the author such as

$$\begin{aligned}\int_{S(R)} |[\tilde{r} \times \tilde{E}] - (k/\omega\epsilon)\tilde{H}|^2 dS = \tilde{0}(R^{-1}) \\ \int_{S(R)} |[\tilde{r} \times \tilde{H}] + (k/\omega\mu)\tilde{E}|^2 dS = \tilde{0}(R^{-1})\end{aligned} \tag{14}$$

instead of (12).

The traditional conditions (11) and (12) require themselves to hold uniformly in all directions. Contrary to this, our conditions (13) and (14) require that (11) and (12), respectively, hold only in the sense of mean of square. Furthermore, our conditions do not require the finiteness of fields like (11) and (12), because the finiteness is automatically fulfilled if (13) or (14) holds [14]. For the proof, see Section 6.11. Therefore, our conditions (13) and (14) are weaker, and hence, better than the conditions (11) and (12).

As will be shown in Sections 6.2 and 6.7, when the condition (13) or (14) is prescribed, the function $I(C(R))$ or $\tilde{I}(S(R))$, which represents the contribution to the fields made by sources located beyond the fictitious boundary $C(R)$ or $S(R)$, that is, the "*residue*" around these singularities, disappears as R tends to infinity. That is, (13) and (14) are the conditions which imply that any source or singularity of the fields does not exist at infinity.

Though the conditions (13) and (14) are a generalization of the traditional conditions (11) and (12), respectively, they are also traditional conditions in the sense that (13) and (14) are not the only radiation conditions but there are infinitely many radiation conditions. In fact in Section 2.9, we shall introduce the generalized radiation conditions by which we shall be able to assign $I(C(R))$ or $\tilde{I}(S(R))$ any field functions which have singularities at infinity. Among them, (13) and (14) are particular conditions which assign the function zero to $I(C(R))$ and $\tilde{I}(S(R))$, respectively.

2.8 Traditional Edge Conditions.

If a boundary has a sharp edge such as a periphery of a disk in a three-dimensional case and end points of a line segment in a two-dimensional case, discussions similar to those on radiation conditions are necessary on edge conditions.

Since Rayleigh (1897) pointed out a possible multiplicity of solutions of a diffraction problem caused by a sharp edge of a boundary, edge conditions, which should be imposed on the behavior of fields in a vicinity of edges so that the diffracted fields are determined uniquely, have been studied by Meixner, Maue, Jones, and Silver and Heins, etc. For the reference to these papers, see [15]. On the basis of the assumption that field energy around an edge should be finite, they obtained their conditions on the local behaviors of fields near an edge, such as order conditions which indicate the order of singularity of fields in a neighborhood of an edge, etc.

We shall not employ these conditions, but employ our own conditions due to the author, which are more simple and clear than the traditional ones, such as

$$\lim \int_{C(\rho)} \{|\partial u/\partial \rho| + |u|\} ds = 0, \qquad (\rho \to 0) \tag{15}$$

for a solution u of the equation (7), and

$$\lim \int_{S(\rho)} \{|\tilde{E}| + |\tilde{H}|\} dS = 0, \qquad (\rho \to 0) \tag{16}$$

for solutions $\tilde{E}$ and $\tilde{H}$ of the equations (4).

It will be shown that these conditions are equivalent to the condition of finite energy. It will also be proved that, when these conditions are imposed, functions $I(C(\rho))$ and $\tilde{I}(S(\rho))$ tend to zero when $\rho \to 0$. For the proof, see Sections 6.12 and 6.13.

Since $I(C(\rho))$ and $\tilde{I}(S(\rho))$ represent the effects to the total fields brought by possible singularities at end points and on edges, or the "*residue*" around the singularities, the last result implies that, under the conditions (15) and (16), there exists no singularity on edges.

It is noted that our conditions (15) and (16) are an improvement of traditional conditions mentioned above, they are still traditional since they are conditions of finite energy. However conditions of finite energy are not the only edge conditions. In fact, in Section 2.9 below, we shall introduce generalized edge conditions by which we see that infinitely many edge conditions are available. From a view point of generalized edge conditions, conditions (15) and (16) are particular ones which assign the function zero to the functions $I(C(\rho))$ and $\tilde{I}(S(\rho))$, respectively.

2.9 Mathematical Model of Fields, Generalized Radiation Conditions and Edge Conditions.

Theoretically, electromagnetic fields in a homogeneous and isotropic medium D are determined if a set of vector functions $\tilde{E}$ and $\tilde{H}$ are obtained so that it satisfies the Maxwell equations (4) everywhere in D except some source points where the fields $\tilde{E}$ and $\tilde{H}$ are generated, and the boundary conditions (8) or (9) on a real boundary as well.

Here we mean by a real boundary, a surface which separates the medium D from other media so that some discontinuity of fields take place there. Contrary to this, we mean by a fictitious boundary a surface imagined in the inside of the medium D such that functions $\tilde{E}$ and $\tilde{H}$ are continuous on and in a vicinity of it.

Excepting some cases of particular geometry of a closed boundary, it is not easy to solve the equations (4) by, say, the method of separation of variables of a partial differential equation, so as to obtain solutions which meet all the requirements mentioned above. However, thanks to the integral representation theorems mentioned in Section 2.5 above, functions $\tilde{E}^t$ and $\tilde{H}^t$, or generally $\tilde{F}^t$ which represent total fields, are given in a necessary and sufficient form to meet the above mentioned requirements except boundary conditions. Therefore, only task left to us is to make $\tilde{E}^t$ and $\tilde{H}^t$ satisfy the boundary conditions, or more precisely, to adjust undetermined densities included in the integrands of the integral represention of $\tilde{E}^t$ and $\tilde{H}^t$ so that boundary conditions are fulfilled.

As was mentioned above in Section 2.5, total fields are given in an abbreviated and unified form as (38); $\tilde{F}^t = \tilde{I}(S + S_0) + \tilde{I}(S^*)$ in the case of a bounded medium, and (39); $\tilde{F}^t = \tilde{I}(S) + \tilde{I}(S^*) + \tilde{I}(S(R))$ in the case of an unbounded medium. If a real boundary has an sharp edge, $\tilde{F}^t$ is supplemented by $\tilde{I}(S(\rho))$ in both cases.

Here, as will be shown by (38), (39) and related expressions in Section 3.2, $\tilde{I}(\Sigma)$'s are integrals taken on a surface element Σ, whose integrands are functions of surface electric and magnetic current densities $\tilde{J}^t = [\tilde{n} \times \tilde{H}^t]$ and $\tilde{J}^{*t} = [\tilde{n} \times \tilde{E}^t]$, and surface electric and magnetic charge densities $\varpi^t = [\tilde{n} \cdot \tilde{E}^t]$ and $\varpi^{*t} = [\tilde{n} \cdot \tilde{H}^t]$ as well. It is noted that these densities have not been determined yet, but should be determined so that $\tilde{F}^t$ satisfy the boundary conditions (8) or (9). It is also noted that (8) and (9) are conditions concerning the densities on a real boundary $S + S_0$ in (38) or S in (39), and not on fictitious boundaries S^*, $S(R)$ and $S(\rho)$. That is, (8) and (9) are the conditions to determine the integrands of integrals $\tilde{I}(S+S_0)$ in (38) and $\tilde{I}(S)$ in (39) only, and we have no way to determine the integrands of integrals $\tilde{I}(S^*)$, $\tilde{I}(S(R))$, and $\tilde{I}(S(\rho))$ so far as we do not assume further conditions about them. Obviously, these conditions must be the ones which reflect the physical situation of the fields. For example, on the basis of Huygens' principle, $\tilde{I}(S^*)$ may be considered to represent the contribution to total fields made by sources located in the inside of $\tilde{S}^*$.

As will be noted by (38) and (39) in Section 3.2, $\tilde{I}(S^*)$ stands for $\tilde{I}_E(S^*) = \tilde{I}_E(P; S^*; \tilde{J}, \tilde{J}^*)$ and $\tilde{I}_H(S^*) = \tilde{I}_H(P; S^*; \tilde{J}, \tilde{J}^*)$ simultaneously. By the same way which reduced (28) to (30), we have

$$\tilde{I}_E(P; S^*; \tilde{J}, \tilde{J}^*) = (1/i\omega\epsilon)\nabla_P \nabla_P \cdot \tilde{A}_P + \nabla_P \times \tilde{A}^*_P - i\omega\mu\tilde{A}_P$$

where

$$(\#) \quad \tilde{A}(P) = \int_{S^*} \tilde{J}_Q \Psi(P,Q) dS_Q$$

$$\tilde{A}^*(P) = \int_{S^*} \tilde{J}^*_Q \Psi(P,Q) dS_Q$$

Similar result is obtained for $\tilde{I}_H(S^*)$ applying the exchange (23). Here, $\tilde{J}(Q)$ and $\tilde{J}^*(Q)$ must be given appropriately corresponding to the property of sources located in S^*, or conversely, we may determine sources which exist in the inside of S^* by providing pertinent values of $\tilde{J}$ and $\tilde{J}^*$ on S^*.

Example

Assume that S^* is a circular cylinder with a radius r and length l, which is parallel to a vector $\tilde{a}$. Suppose that $\tilde{J}^* = 0$, $\tilde{J} = J\tilde{a}$. Then, in the limit as r tends to zero, we have $\tilde{A}(P) = I\Psi(P,Q^*)\tilde{a}$ where Q^* is the center of the cylinder and $I = 2\pi r l J$. This is the well known vector potential due to an electric dipole with the current density $I\tilde{a}$ located at Q^*.

The result held for $\tilde{I}(S^*)$ above also holds for both $\tilde{I}(S(R))$ and $\tilde{I}(S(\rho))$. That is, vector potentials for $\tilde{I}(S(R))$ are obtained if the range of integration S^* in (#) is replaced by $S(R)$. Similarly, if S^* is replaced by $S(\rho)$, then (#) defines the potentials for $\tilde{I}(S(\rho))$. Furthermore, in order to make $\tilde{I}(S(R))$ definite, we must give $\tilde{J}$ and $\tilde{J}^*$ on $S(R)$ so that they reflect the property of singularities at infinity, and conversely, possible singularities which exist beyond $S(R)$, or at least the contributions of them to total fields, are determined by prescribing $\tilde{J}$ and $\tilde{J}^*$ on $S(R)$. Similarly, to determine singularities induced on the periphery of sharp edges of a real boundary is equivalent to prescribe $\tilde{J}$ and $\tilde{J}^*$ on $S(\rho)$. We call conditions to prescribe $\tilde{J}$ and $\tilde{J}^*$ on $S(R)$, or what is the same thing, conditions to make $\tilde{I}(S(R))$ known functions which satisfy the Maxwell equations (4) everywhere in the whole space except at points at infinity, generalized radiation conditions. Also, we call conditions to prescribe $\tilde{J}$ and $\tilde{J}^*$ on $S(\rho)$, or what is the same thing, conditions to make $\tilde{I}(S(\rho))$ known functions which satisfy (4) everywhere in the whole space except at points on the edges of a boundary, generalized edge conditions.

Because there are infinitely many selections of $\tilde{J}$ and $\tilde{J}^*$, there are infinitely many radiation conditions and edge conditions. However, what is important is that solutions of the equations (4) are uniquely determined for each one of arbitrary selections of $\tilde{I}(S(R))$ and $\tilde{I}(S(\rho))$. For details, see Section 6.11.

As will be proved in Section 6.7, when the traditional condition (14) is applied, $\tilde{I}(S(R))$ vanish. That is, (14) is a particular radiation condition which prescribes the function zero to $\tilde{I}(S(R))$. When $\tilde{I}(S(R))$ vanish, expressions (39) mentioned above are reduced to $\tilde{F}^t(P) = \tilde{I}(S) + \tilde{I}(S^*)$, which satisfy the Maxwell equations (4) everywhere in the exterior unbounded domain outside of S. That is, (14) is also the condition which prove that there is no singularity, or source, of fields at infinity.

As will be shown in Section 6.13, the traditional edge condition (16) makes $\tilde{I}(S(\rho))$ zero. That is, (16) is a particular edge condition which prescribes the function zero to $\tilde{I}(S(\rho))$, or is the condition which guarantees that no singularity has been induced on the edge of a boundary.

When some radiation conditions and edge conditions are applied, primary fields $\tilde{I}(S^*)$, $\tilde{I}(S(R))$ and $\tilde{I}(S(\rho))$ become known, while only secondary fields $\tilde{I}(S+S_0)$ in the case of bounded medium, and $\tilde{I}(S)$ in the case of an unbounded medium, remain undetermined, which are to be determined so that they make the total fields satisfy the boundary conditions (8) or (9). Thus the problem to determine electromagnetic fields has been reduced to a boundary value problem concerning secondary fields under certain radiation, edge and boundary conditions. It is noted that, by the uniqueness theorem which will be given in Section 6.11, electromagnetic fields are determined uniquely corresponding to each choice of radiation conditions and edge conditions.

Among other approaches to the boundary value problems, we shall study approaches by integral equations in Section 6.10, and approaches by series expansion in Section 5.

Next, we shall study the case of a two-dimensional E wave. Because the situation is quite similar between two and three dimensional fields, we shall make brief discussions along the thread of study on three-dimensional fields mentioned above.

A two-dimensional E wave is determined by a solution of the Helmholtz equation (7) which satisfies the boundary condition (10) or (10)'. On the other hand, thanks to the integral representation formulas, a solution has already been given by $(18)'$; $u^t(P) = I(C + C_0) + I(C^*)$ in the case of a bounded medium, and by $(19)'$; $u^t(P) = I(C) + I(C^*) + I(C(R))$ in the case of an unbounded medium. Here, $I(\mathcal{L})$'s are certain integrals on curve $\mathcal{L}$ indicated in parenthesis, whose integrands being functions of $\sigma^t = u^t$ and $\tau^t = \partial u^t/\partial n$. For details, see Section 3.1.

Therefore, only task left to us is to make u^t satisfy the boundary condition, or more precisely, to make σ^t and τ^t included in the integral $(18)'$ or $(19)'$ so that (10) or $(10)'$ is met.

Among others, $I(C + C_0)$ in $(18)'$, and $I(C)$ in $(19)'$, are the subject of a boundary value problem with the condition (10) or $(10)'$, because (10) and $(10)'$ are the conditions on the density on a real boundary C and C_0. Contrary to them, we have no condition to determine σ and τ on C^*, $C(R)$ and $C(\rho)$. Hence, we have to assign appropriate values to them so that the physical properties of $I(C)$, $I(C(R))$ and $I(C(\rho))$ are well described. We call the condition to specify densities on $C(R)$ generalized radiation condition, while the one to specify densities on $C(\rho)$ generalized edge condition. Obviously, there are infinitely many radiation conditions and edge conditions which make a solution of (7) unique. When some radiation condition and edge condition are applied, $I(C(R))$ and $I(C(\rho))$ become known, and we are left with boundary value problems to solve $I(C+C_0)$ and $I(C)$, the details of which will be studied in Sections 5 and 6.10.

2.10 Eigen Fields in an Unbounded Domain.

When there is no primary incident field, corresponding non-trivial fields, if exist, are called eigen fields. It is known that eigen fields may exist in a bounded closed domain. In this section, we shall give a discussion on the existence of eigen fields in an unbounded unclosed domain. In the discussion, we shall make use of the uniqueness theorem which will be proved in Section 6.11. According to the theorem, it is shown that functions $\tilde{E}$ and $\tilde{H}$, which satisfy the Maxwell equations (4) and the radiation conditions (14) in an unbounded domain D in the exterior of a surface S, vanish everywhere in D if they assume a boundary value $[\tilde{n}\times\tilde{E}]\cdot\overline{\tilde{H}}=0$ on S, where $\overline{\tilde{H}}$ indicates the complex conjugate of $\tilde{H}$. Also, if $[\tilde{n}\times\tilde{E}]\cdot\overline{\tilde{H}}\neq 0$, then $\tilde{E}$ and $\tilde{H}$ in D do not vanish. From (39) in Section 3.2, we have $\tilde{I}(S)=\tilde{F}^t(P)-\tilde{I}(S^*)-\tilde{I}(S(R))$, where $\tilde{I}(S^*)$ are given functions representing incident fields caused by sources, while $\tilde{I}(S(R))$ are also known functions given by generalized radiation conditions. Since this expression is a unification of two expressions in $(37)'$, we have $(*)$ $\tilde{I}_E(S)=\tilde{E}^t(P)-\tilde{I}_E(S^*)-\tilde{I}_E(S(R))$, which we shall set $=\tilde{E}(P)$. We also have similar expression for $\tilde{H}(P)=\tilde{I}_H(S)$. Furthermore, as will be proved in Section 6.9, these functions $\tilde{E}(P)$ and $\tilde{H}(P)$ satisfy the Maxwell equations and the radiation conditions (14) in the unbounded unclosed domain exterior to S. On the other hand, from $(*)$, we have $[\tilde{n}\times\tilde{E}]=[\tilde{n}\times\tilde{E}^t]-[\tilde{n}\times\tilde{I}_E(S^*)]-[\tilde{n}\times\tilde{I}_E(S(R))]$

If there is no source in the interior of S^*, then $\tilde{I}_E(S^*)=0$. Furthermore, if the radiation conditions (14) are employed, then $\tilde{I}_E(S(R))=0$. Hence, we have $[\tilde{n}\times\tilde{E}]=[\tilde{n}\times\tilde{E}^t]$, which turns out to be $[\tilde{n}\times\tilde{E}]=0$ because of the boundary condition (9); $[\tilde{n}\times\tilde{E}^t]=0$. Consequently, by virtue of the uniqueness theorem mentioned above, $\tilde{E}$, and hence $\tilde{H}$, are zero everywhere in the domain exterior to S. That is, no eigen fields can exist in an unbounded unclosed domain as far as the radiation conditions (14) are employed. However, if another radiation condition is employed, we have $[\tilde{n} \times \tilde{E}] = -[\tilde{n} \times \tilde{I}_E(S(R))] \neq 0$ from which we see that non-trivial fields can exist even if there are no primary incident fields $\tilde{I}_E(S^*)$. Thus we have shown that the possibility of existence of eigen fields depends on the choice of radiation conditions.

3. Integral Representation Formulas of Electromagnetic Fields (Case of a Closed Boundary)

In this and the next sections, integral representation formulas of electromagnetic fields will be introduced. They will play an important role in the present work, because the author's theory presented here is fully based on them.

It is noted that the formulas obtained here are of a necessary and sufficient form to represent electromagnetic fields in the sense that any fields are necessarily represented by these formulas and conversely, functions given by the formulas satisfy all requirements which make them really electromagnetic fields.

In this Section 3, formulas for fields in a domain surrounded by a closed boundary are studied, while in the next Section 4, formulas for fields in a domain with an unclosed boundary will be obtained.

3.1 Representation of a Solution of the Helmholtz Equation.

Suppose that C is a union of closed and piecewise smooth contours $C_1, \cdots, C_N$ in a plane such that each one of which exists in the outside of the others, that C^* is a closed curve enclosing source points of a field u, and that C_0 is a closed contour enclosing all of C and C^*. Let D be the bounded domain surrounded by C, C^* and C_0, and $\tilde{n}$ be the normal vector on C, C^* and C_0 directed outward of D. Let us denote points by P and Q, their distance by $\overline{PQ}$, and a function u evaluated at P by $u(P)$, etc. We introduce the two-dimensional free space Green function as follows,

$$\Psi = \Psi(P, Q) = (1/4i) H_0^{(2)}(\kappa R) \tag{17}$$

Here, $R = \overline{PQ}$, κ is a constant such as $\mathrm{Im}\kappa \leq 0$, and $H_0^{(2)}(\kappa R)$ is the zero-th order Hankel function of the second kind. We employ a symbol $\in$ such as $P \in C$ means that P is a point on C, while $P \notin C$ means that P does not exist on C. Now, we have the following theorem.

Assume that there exists a solution to the two dimensional Helmholtz equation (7) in a domain D, which we shall describe as u^t, where superscript t indicates a total field, then it is necessarily represented by

$$u^t(P) = \int_{C+C^*+C_0} [\Psi(P,Q)\tau^t(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma^t(Q)]ds_Q \tag{18}$$

when $P \in D$, where σ^t and τ^t are boundary values of $u^t(Q)$ and $\partial u^t(Q)/\partial n(Q)$, respectively.

On the other hand, if $P \notin D + C + C^* + C_0$, the function defined by (18) is identically zero.

Conversely, a function defined by the integral in the right hand side of (18), whose integrand being given in terms of arbitrary integrable functions σ and τ, satisfies the equation (7) in D. Thus we have shown that a solution of (7) is equivalent to the function given by the right member of (18). That is, (18) represents a solution of the Helmholtz equation, or of a field component $u^t = E_z$, in a bounded domain D, and there is no other solution besides that given by (18). The proof of (18) is given in Section 6.1.

Next, let us study a case where a domain D is unbounded.

Suppose that C is a union of closed contours $C_1, \cdots, C_N$, and that C^* is a closed curve enclosing source points, both of which have been introduced above. Let D be the exterior of C and C^*, which is hence unbounded.

We enclose both C and C^* by a circle $C(R)$ of a sufficiently large radius R. Then, (18) remains to be true if C_0 is replaced by $C(R)$, and we see that a solution of (7) in an unbounded domain D, if exists, is necessarily represented as

$$u^t(P) = \int_{C+C^*+C(R)} [\Psi(P,Q)\tau^t(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma^t(Q)]ds_Q \tag{19}$$

when $P \in D$. While, when $P \notin D + C + C^* + C(R)$, the function defined by the right member of (19) is identically zero.

Conversely, the right member of (19) defined in terms of arbitrary densities σ and τ, satisfies (7). Thus, a solution of (7) in D and a function defined by (19) have been shown to be equivalent to each other. That is, (19) represents a solution of (7) in an unbounded domain D, and there in no other solution besides that given by (19).

For the sake of simplicity, we shall put

$$I(P;\mathcal{L}) = \int_{\mathcal{L}} [\Psi(P,Q)\tau^t(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma^t(Q)]ds_Q \qquad (20)$$

then, (18) is rewritten as

$$u^t(P) = I(P;C+C_0) + I(P;C^*) \qquad (18)'$$

while (19) is rewritten as

$$u^t(P) = I(P;C) + I(P;C^*) + I(P;C(R)) \qquad (19)'$$

If more brevity is preferable, $I(P;\mathcal{L})$ will be written as $I(\mathcal{L})$, which notations have been used in Section 2.5.

From the point of view of physics, these expressions are a mathematical version of Huygens' principle. For instance, $I(P;C^*)$ and $I(P;C(R))$, respectively, denote contributions to the total field $u^t(P)$ made by sources located in the inside of C^* and outside of $C(R)$. It will be proved in Section 6.2 that $I(P;C(R))$ tends to zero if the radiation condition (13) is prescribed and if R tends to infinity, showing that there is no source at infinity.

Set $u(P) = u^t(P) - I(P;C^*) - I(P;C(R))$, then, from (19) and (19)$'$, we have

$$u(P) = I(P;C) = \int_C [\Psi(P,Q)\tau^t(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma^t(Q)]ds_Q \qquad (21)$$

where, as was mentioned above, σ^t and τ^t are the boundary values of the function u^t. However, as will be proved in Section 6.3, $u(P)$ is reduced to

$$u(P) = I(P;C) = \int_C [\Psi(P,Q)\tau(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma(Q)]ds_Q \qquad (22)$$

where σ and τ are the boundary values of u itself and not of u^t. Furthermore, $u(P)$ defined by (22) in terms of arbitrary σ and τ

satisfies the Helmholtz equation (7) everywhere in the exterior of C, and the radiation condition (13) at infinity as well. For the proof, see Section 6.4.

Thus we have introduced the formula (18) or (18)$'$ for the case of a bounded domain, and (19) or (19)$'$ for the case of an unbounded domain, which represents a solution of the Helmholtz equation (7) uniquely so that there is no other solution besides them. Though these results are concerned with the case where a boundary is a closed curve, they will serve to obtain formulas for a case of an unclosed boundary in Section 4. These formulas will be the basis of our theory of a two-dimensional field.

As was discussed in Section 2.9, $I(P;C^*)$ and $I(P;C(R))$ are arbitrarily given functions representing known primary field brought by sources located in the inside of C^* and outside of $C(R)$, respectively. While $u(P)=I(P;C)$ is a secondary field which should be determined so as to make u^t satisfy boundary conditions prescribed on C.

3.2 Representation of Electromagnetic Fields.

Suppose that S is a union of closed and piecewise smooth surfaces $S_1,\cdots,S_N$, such that each one of which exists in the outside of the others, that S^* is a closed surface enclosing source points where electromagnetic fields are generated, and that S_0 is a surface enclosing all of S and S^*. Assume that D is the bounded domain surrounded by S, S^* and S_0, and that $\tilde{n}$ is the normal on S, S^* and S_0, directed inward of D. To begin with, analogously to the preceding section, we shall study the integral representation formulation of solutions of Maxwell's equations (4) in a bounded domain D mentioned above.

First of all, it is noted that the equations in (4) are invariant as a whole by the exchange

$$E \rightleftharpoons H \qquad \epsilon \rightleftharpoons -\mu \tag{23}$$

Therefore, a result derived from (4) has always its counterpart obtained from it by the exchange (23).

Next, we introduce the free space Green function

$$\Psi=\Psi(P,Q)=e^{-ikR}/4\pi R, \quad R=\overline{PQ}$$

where (6), i.e.

$$k^2 = \omega^2 \epsilon \mu, \qquad Im.k \leq 0$$

and also the following functions

$$\begin{aligned}\tilde{K}_E(P,Q;\varpi,\varpi^*,\tilde{J},\tilde{J}^*) =&\varpi(Q)\nabla_Q\Psi(P,Q)+\\ &\tilde{J}^*(Q)\times\nabla_Q\Psi(P,Q) - i\omega\mu\Psi(P,Q)\tilde{J}(Q)\\ \tilde{K}_H(P,Q;\varpi,\varpi^*,\tilde{J},\tilde{J}^*) =&\varpi^*(Q)\nabla_Q\Psi(P,Q)+\\ &\tilde{J}(Q)\times\nabla_Q\Psi(P,Q) + i\omega\epsilon\Psi(P,Q)\tilde{J}^*(Q)\end{aligned} \tag{24}$$

where $\varpi(Q)$, $\varpi^*(Q)$, $\tilde{J}(Q)$ and $\tilde{J}^*(Q)$ are integrable functions of a point Q, while ∇_Q is the nabla operator operated at Q.

If a solution $\tilde{E}^t$ and $\tilde{H}^t$ of the Maxwell equations (4) exists, where superscript t indicates total fields, it should necessarily be represented as

$$\begin{aligned}\tilde{E}^t(P) &= \int_{S+S^*+S_0} \tilde{K}_E(P,Q;\varpi^t,\varpi^{*t},\hat{J}^t,\hat{J}^{*t})dS_Q\\ \tilde{H}^t(P) &= \int_{S+S^*+S_0} \tilde{K}_H(P,Q;\varpi^t,\varpi^{*t},\tilde{J}^t,\tilde{J}^{*t})dS_Q\end{aligned} \tag{25}$$

when $P \in D$, where $\tilde{K}_E$ and $\tilde{K}_H$ are defined by (24) with ϖ, ϖ^*, $\tilde{J}$ and $\tilde{J}^*$ replaced by $\varpi^t = (\tilde{n}\cdot\tilde{E}^t)$, $\varpi^{*t} = (\tilde{n}\cdot\tilde{H}^t)$, $\tilde{J}^t = [\tilde{n}\times\tilde{H}^t]$ and $\tilde{J}^{*t} = [\tilde{n}\times\tilde{E}^t]$, respectively. Here, respectively, ϖ^t and ϖ^{*t} are surface electric and magnetic charge densities, while $\tilde{J}^t$ and $\tilde{J}^{*t}$ are surface electric and magnetic current densities, induced on the boundaries S, S^* and S_0 by the total fields $\tilde{E}^t$ and $\tilde{H}^t$.

When $P \notin D+S+S^*+S_0$, the functions defined by the right member of (25) are identically zero. Proof of (25) is given in Section 6.5.

The expressions (25) are the well known Stratton-Chu Formula. However, it should be noted that, though a solution of the Maxwell equations is necessarily represented by (25), functions defined by (25) in terms of arbitrary functions ϖ^t, ϖ^{*t}, $\tilde{J}^t$ and $\tilde{J}^{*t}$ do not necessarily satisfy the Maxwell equations. This is easily proved by elementary calculations. In other words, expressions (25) include extra functions besides electromagnetic field functions. Hence, if one depends on (25) as the basis of his analysis, he may suffer from spurious solutions in his result.

In order to make the expressions necessary and sufficient one so that we shall meet no spurious solution, we should have some additional relations between surface current and surface charge densities. These are essentially the continuity relations between them, which are given by the following theorem.

Assume that ϖ, ϖ^*, $\tilde{J}$ and $\tilde{J}^*$, respectively, are the boundary values $(\tilde{n}\cdot\tilde{E})$, $(\tilde{n}\cdot\tilde{H})$, $[\tilde{n}\times\tilde{H}]$ and $[\tilde{n}\times\tilde{E}]$ of a solution $\tilde{E}$ and $\tilde{H}$ of the equations (4), then, we can prove, for $P\notin S+S^*+S_0$, that

$$\begin{aligned}\int_{S+S^*+S_0}\tilde{J}_Q\cdot\nabla_Q\Psi(P,Q)dS_Q &= \\ &i\omega\epsilon\int_{S+S^*+S_0}\varpi_Q\Psi(P,Q)dS_Q \\ \int_{S+S^*+S_0}\tilde{J}^*_Q\cdot\nabla_Q\Psi(P,Q)dS_Q &= \\ &-i\omega\mu\int_{S+S^*+S_0}\varpi^*_Q\Psi(P,Q)dS_Q\end{aligned} \tag{26}$$

and

$$\begin{aligned}\int_{S+S^*+S_0}(\tilde{J}_Q\cdot\nabla_Q)\nabla_Q\Psi(P,Q)dS_Q &= \\ &i\omega\epsilon\int_{S+S^*+S_0}\varpi_Q\nabla_Q\Psi(P,Q)dS_Q \\ \int_{S+S^*+S_0}(\tilde{J}^*_Q\cdot\nabla_Q)\nabla_Q\Psi(P,Q)dS_Q &= \\ &-i\omega\mu\int_{S+S^*+S_0}\varpi^*_Q\nabla_Q\Psi(P,Q)dS_Q\end{aligned} \tag{27}$$

Furthermore, (26) and (27) are equivalent to each other. The proof of (26) and (27) is given in Section 6.6.

It is easily shown, by an elementary calculation, that fields $\tilde{E}$ and $\tilde{H}$ represented by (25) satisfy the Maxwell equations, if (27) is taken into consideration.

Conversely, if ϖ, ϖ^*, $\tilde{J}$ and $\tilde{J}^*$ are integrable functions which satisfy the relation (26) but otherwise arbitrary, then, $\tilde{E}$ and $\tilde{H}$ defined by (25) in terms of these functions are easily proved to satisfy the Maxwell equations. That is, functions $\tilde{E}$ and $\tilde{H}$ defined by (25) and (26) make a solution of the Maxwell equations (4), and there is no other solution of (4) besides that given by (25) and (26).

Furthermore, on substituting (27) in (25), we have

$$\begin{aligned}\tilde{E}^t(P) &= \int_{S+S^*+S_0} \tilde{G}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q \\ \tilde{H}^t(P) &= \int_{S+S^*+S_0} \tilde{G}_H(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q\end{aligned} \tag{28}$$

where we have set

$$\begin{aligned}\tilde{G}_E(P,Q;\tilde{J},\tilde{J}^*) =&(1/i\omega\epsilon)(\tilde{J}_Q\cdot\nabla_Q)\nabla_Q\Psi(P,Q) \\ &+\tilde{J}_Q^*\times\nabla_Q\Psi(P,Q)-i\omega\mu\Psi(P,Q)\tilde{J}_Q \\ \tilde{G}_H(P,Q;\tilde{J},\tilde{J}^*) =&(i/\omega\mu)(\tilde{J}_Q^*\cdot\nabla_Q)\nabla_Q\Psi(P,Q) \\ &+\tilde{J}_Q\times\nabla_Q\Psi(P,Q)+i\omega\epsilon\Psi(P,Q)\tilde{J}_Q^*\end{aligned} \tag{29}$$

and $\tilde{J}^t=[\tilde{n}\times\tilde{H}^t]$, $\tilde{J}^{*t}=[\tilde{n}\times\tilde{E}^t]$.

Note that (28) was derived in [16] also in a different form.

The functions defined by (28) are simpler, and hence better, than those defined by (25), because they are given in terms of arbitrary tangential vectors $\tilde{J}$ and $\tilde{J}^*$ only, without referring to ϖ and ϖ^*. Furthermore, these functions given by (28) are easily proved to satisfy the Maxwell equations (4).

As a consequence, we have proved that the set of expressions (25) and (27), or the expressions (28), furnish the complete representation formulas of electromagnetic fields in a bounded domain D.

The formulas (28) can easily be rewritten as

$$\begin{aligned}\tilde{E}^t(P) =&(1/i\omega\epsilon)\nabla_P\nabla_P\cdot\tilde{A}_P+\nabla_P\times\tilde{A}_P^*-i\omega\mu\tilde{A}_P \\ \tilde{H}^t(P) =&(i/\omega\mu)\nabla_P\nabla_P\cdot\tilde{A}_P^*+\nabla_P\times\tilde{A}_P+i\omega\epsilon\tilde{A}_P^*\end{aligned} \tag{30}$$

where $\tilde{A}_P$ and $\tilde{A}_P^*$ are vector potentials defined by

$$\begin{aligned}\tilde{A}_P = \tilde{A}(P) &= \int_{S+S^*+S_0} \tilde{J}_Q^t\Psi(P,Q)dS_Q \\ \tilde{A}_P^* = \tilde{A}^*(P) &= \int_{S+S^*+S_0} \tilde{J}_Q^{*t}\Psi(P,Q)dS_Q\end{aligned} \tag{31}$$

Also, if we employ the following notations

$$\begin{aligned}\tilde{I}_E(P;\Sigma;\tilde{J},\tilde{J}^*) &= \int_\Sigma \tilde{G}_E(P,Q;\tilde{J},\tilde{J}^*)dS_Q \\ \tilde{I}_H(P;\Sigma;\tilde{J},\tilde{J}^*) &= \int_\Sigma \tilde{G}_H(P,Q;\tilde{J},\tilde{J}^*)dS_Q\end{aligned} \tag{32}$$

(28) is rewritten as

$$\begin{aligned}\tilde{E}^t(P) &= \tilde{I}_E(P; S+S_0; \tilde{J}^t, \tilde{J}^{*t}) + \tilde{I}_E(P; S^*; \tilde{J}^t, \tilde{J}^{*t}) \\ \tilde{H}^t(P) &= \tilde{I}_H(P; S+S_0; \tilde{J}^t, \tilde{J}^{*t}) + \tilde{I}_H(P; S^*; \tilde{J}^t, \tilde{J}^{*t})\end{aligned} \tag{33}$$

which, for the sake of simplicity, may be rewritten as

$$\begin{aligned}\tilde{E}^t(P) &= \tilde{I}_E(S+S_0) + \tilde{I}_E(S^*) \\ \tilde{H}^t(P) &= \tilde{I}_H(S+S_0) + \tilde{I}_H(S^*)\end{aligned} \tag{33$'$}$$

By an elementary calculation, it is easy to see that the pair of the right member of expressions (33) satisfies the Maxwell equations (4), that is, it makes a set of electromagnetic fields. Furthermore, not only this pair, but also each one of the pairs $\{\tilde{E}, \tilde{H}\} = \{\tilde{I}_E(S+S_0), \tilde{I}_H(S+S_0)\}$ and $\{\tilde{E}^*, \tilde{H}^*\} = \{\tilde{I}_E(S^*), \tilde{I}_H(S^*)\}$ makes electromagnetic fields. In other words, total fields $\{\tilde{E}^t, \tilde{H}^t\}$ are composed of these partial fields. Obviously, the first partial fields are functions defined by integrals on the real boundary $S+S_0$, which separates the medium where the total fields are considered from other media, while the second partial fields are given by integrals on a fictitious boundary S^* surrounding source points.

Next, we shall study representation formulas of electromagnetic fields in an unbounded domain. As in the above, we denote by S a union of closed surfaces $S_1, \cdots, S_N$, and by S^* a closed surface enclosing source points. We consider the domain exterior to S and S^*, which is hence unbounded. Representation formulas for fields at a point P in an unbounded domain are given by the results obtained above for a bounded domain, provided S_0 is replaced by a sphere $S(R)$ of a sufficiently large radius R, which encloses surfaces S and S^*, and a point of observation P as well. That is, corresponding to (25), we have

$$\begin{aligned}\tilde{E}^t(P) &= \int_{S+S^*+S(R)} \tilde{K}_E(P, Q; \varpi^t, \varpi^{*t}, \tilde{J}^t, \tilde{J}^{*t}) dS_Q \\ \tilde{H}^t(P) &= \int_{S+S^*+S(R)} \tilde{K}_H(P, Q; \varpi^t, \varpi^{*t}, \tilde{J}^t, \tilde{J}^{*t}) dS_Q\end{aligned} \tag{34}$$

and, corresponding to (26), we have

$$\begin{aligned}\int_{S+S^*+S(R)} \tilde{J}_Q^t \cdot \nabla_Q \Psi(P,Q) dS_Q = \\ i\omega\epsilon \int_{S+S^*+S(R)} \varpi_Q^t \Psi(P,Q) dS_Q \\ \int_{S+S^*+S(R)} \tilde{J}_Q^{*t} \cdot \nabla_Q \Psi(P,Q) dS_Q = \\ -i\omega\mu \int_{S+S^*+S(R)} \varpi_Q^{*t} \Psi(P,Q) dS_Q\end{aligned} \tag{35}$$

Also, corresponding to (28), we have

$$\begin{aligned}\tilde{E}^t(P) &= \int_{S+S^*+S(R)} \tilde{G}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t}) dS_Q \\ \tilde{H}^t(P) &= \int_{S+S^*+S(R)} \tilde{G}_H(P,Q;\tilde{J}^t,\tilde{J}^{*t}) dS_Q\end{aligned} \tag{36}$$

which are, corresponding to (33), rewritten as

$$\begin{aligned}\tilde{E}^t(P) = \tilde{I}_E(P;S;\tilde{J}^t,\tilde{J}^{*t}) + \tilde{I}_E(P;S^*;\tilde{J}^t,\tilde{J}^{*t}) + \\ \tilde{I}_E(P;S(R);\tilde{J}^t,\tilde{J}^{*t}) \\ \tilde{H}^t(P) = \tilde{I}_H(P;S;\tilde{J}^t,\tilde{J}^{*t}) + \tilde{I}_H(P;S^*;\tilde{J}^t,\tilde{J}^{*t}) + \\ \tilde{I}_H(P;S(R);\tilde{J}^t,\tilde{J}^{*t})\end{aligned} \tag{37}$$

For the sake of simplicity, (37) may be rewritten as

$$\begin{aligned}\tilde{E}^t(P) &= \tilde{I}_E(S) + \tilde{I}_E(S^*) + \tilde{I}_E(S(R)) \\ \tilde{H}^t(P) &= \tilde{I}_H(S) + \tilde{I}_H(S^*) + \tilde{I}_H(S(R))\end{aligned} \tag{37$'$}$$

Similar to what was mentioned about (33), total fields $\{\tilde{E}^t, \tilde{H}^t\}$ given by (37) are easily shown to be composed of three partial fields $\{\tilde{E}, \tilde{H}\} = \{\tilde{I}_E(S), \tilde{I}_H(S)\}$, $\{\tilde{E}^*, \tilde{H}^*\} = \{\tilde{I}_E(S^*), \tilde{I}_H(S^*)\}$ and $\{\tilde{E}_R, \tilde{H}_R\} = \{\tilde{I}_E(S(R)), \tilde{I}_H(S(R))\}$, where the first partial fields are defined by integrals on the real boundary S, while the second and third partial fields are given by integrals on fictitious boundaries S^* and $S(R)$, respectively.

Thus we have proved that the expressions (34) together with (35), or the expressions (36), or the expressions (37), furnish us with the complete representation formulas for electromagnetic fields in an unbounded domain. As will be proved in Section 6.7, $\tilde{I}_E(P;S(R);\tilde{J}^t,\tilde{J}^{*t})$ and $\tilde{I}_H(P;S(R);\tilde{J}^t,\tilde{J}^{*t})$, which appear in (37), vanish when R tends to infinity, provided the radiation conditions (14) are imposed on $\tilde{E}^t$ and $\tilde{H}^t$

For the sake of simplicity, we write $\tilde{I}(P;\Sigma;\tilde{J},\tilde{J}^*)$ generally to denote $\tilde{I}_E(P;\Sigma;\tilde{J},\tilde{J}^*)$ and $\tilde{I}_H(P;\Sigma;\tilde{J},\tilde{J}^*)$, $\tilde{F}(P)$ to denote $\tilde{E}(P)$ and $\tilde{H}(P)$, and $\tilde{G}(P,Q;\tilde{J},\tilde{J}^*)$ to denote $\tilde{G}_E(P,Q;\tilde{J},\tilde{J}^*)$ and $\tilde{G}_H(P,Q;\tilde{J},\tilde{J}^*)$.

Then, the two expressions in (33) may be unified and written generally as

$$\tilde{F}^t(P) = \tilde{I}(P;S+S_0;\tilde{J}^t,\tilde{J}^{*t}) + \tilde{I}(P;S^*;\tilde{J}^t,\tilde{J}^{*t}) \tag{38}$$

Similarly, the two expressions in (37) may be unified and written as

$$\tilde{F}^t(P) = \tilde{I}(P;S;\tilde{J}^t,\tilde{J}^{*t}) + \tilde{I}(P;S^*;\tilde{J}^t,\tilde{J}^{*t}) + \tilde{I}(P;S(R);\tilde{J}^t,\tilde{J}^{*t}) \tag{39}$$

If more brevity is preferable, $\tilde{I}(P;\Sigma;\tilde{J},\tilde{J}^*)$ can be written as $\tilde{I}(\Sigma)$, which notation has been used in (33)$'$, (37)$'$ and in Section 2.5.

From (39), we have

$$\tilde{F}(P) = \tilde{F}^t(P) - \tilde{I}(P;S^*;\tilde{J}^t,\tilde{J}^{*t}) - \tilde{I}(P;S(R);\tilde{J}^t,\tilde{J}^{*t}) = \tilde{I}(P;S;\tilde{J}^t,\tilde{J}^{*t})$$

which is, by (32),

$$\tilde{F}(P) = \tilde{I}(P;S;\tilde{J}^t,\tilde{J}^{*t}) = \int_S \tilde{G}(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q \tag{40}$$

where $\tilde{G}(P,Q;\tilde{J}^t,\tilde{J}^{*t})$, that is $\tilde{G}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t})$ and $\tilde{G}_H(P,Q;\tilde{J}^t,\tilde{J}^{*t})$, have been defined by (29) in terms of the tangential components of total fields $\tilde{J}^t = [\tilde{n}\times\tilde{H}^t]$ and $\tilde{J}^{*t} = [\tilde{n}\times\tilde{E}^t]$. However, by a way similar to that derived (22) from (21), we can replace $\tilde{J}^t$ and $\tilde{J}^{*t}$ by $\tilde{J} = [\tilde{n}\times\tilde{H}]$ and $\tilde{J}^* = [\tilde{n}\times\tilde{E}]$, which are not the tangential components of total fields $\tilde{E}^t$ and $\tilde{H}^t$, but are the tangential components of secondary fields $\tilde{E}$ and $\tilde{H}$ themselves. That is, we have, instead of (40),

$$\tilde{F}(P) = \tilde{I}(P;S;\tilde{J},\tilde{J}^*) = \int_S \tilde{G}(P,Q;\tilde{J},\tilde{J}^*)dS_Q \tag{41}$$

or

$$\tilde{E}(P) = \tilde{I}_E(P; S; \tilde{J}, \tilde{J}^*) = \int_S \tilde{G}_E(P, Q; \tilde{J}, \tilde{J}^*) dS_Q$$
$$\tilde{H}(P) = \tilde{I}_H(P; S; \tilde{J}, \tilde{J}^*) = \int_S \tilde{G}_H(P, Q; \tilde{J}, \tilde{J}^*) dS_Q \qquad (42)$$

For the proof, see Section 6.8.

Furthermore, it should be noted that $\tilde{E}(P)$ and $\tilde{H}(P)$ defined by (42) satisfy the Maxwell equations everywhere in the exterior of S, and also the radiation condition (14) at infinity. For the proof, see Section 6.9.

The same holds in a case of a closed boundary as well. That is, if we consider (38) instead of (39) and if the range of integration of $\tilde{F}(P)$ in (40) is replaced by $S + S_0$, then we can prove (41) where S is replaced by $S + S_0$.

Now we conclude this section as follows. We have obtained, as the fundamental integral representation formulas, the set of expressions (25) and (26), or the expressions (28), for electromagnetic fields in a bounded domain surrounded by closed surfaces S, S^* and S_0. On the other hand, we have introduced the set of expressions (34) and (35), or the expressions (36), as the fundamental representation formulas for electromagnetic fields in an unbounded domain exterior to closed surfaces S and S^*.

The expressions (28) were rewritten as the two expressions in (33), which were then unified into a single expression in (38). Similarly, the expressions (36) were rewritten as the two expressions in (37), which were then unified into a single expression in (39).

These formulas furnish us with the necessary and sufficient formulation of electromagnetic fields in the sense that any solutions of the Maxwell equations are necessarily represented by these formulas, and conversely, functions given by these formulas do satisfy the Maxwell equations. In other words, the functions defined by these formulas determine electromagnetic fields, and there are no other fields besides them.

From the point of view of physics, these formulas are a mathematical version of the Huygens principle.

As was shown by (33)$'$ and the following note, total fields $\{\tilde{E}^t, \tilde{H}^t\}$ in a bounded domain were decomposed to the sum of partial fields $\{\tilde{E}, \tilde{H}\}$ and $\{\tilde{E}^*, \tilde{H}^*\}$, while total fields $\{\tilde{E}^t, \tilde{H}^t\}$ in an unbounded domain were shown by (37)$'$ and the following note to be the sum of partial fields $\{\tilde{E},\tilde{H}\}$, $\{\tilde{E}^*,\tilde{H}^*\}$ and $\{\tilde{E}_R,\tilde{H}_R\}$.

Among these partial fields, $\{\tilde{E}^*,\tilde{H}^*\}$, which are defined by integrals on a fictitious boundary S^* surrounding source points of total fields, are to be given so as to be the incident fields corresponding to the sources. And $\{\tilde{E}_R,\tilde{H}_R\}$ in $(37)'$ are also to be given, by the generalized radiation conditions studied in Section 2.9, so as to represent the effect of singularity located at infinity. Finally, $\{\tilde{E}+\tilde{E}_0,\tilde{H}+\tilde{H}_0\}$ in $(33)'$ or $\{\tilde{E},\tilde{H}\}$ in $(37)'$ are secondary fields which, as was studied in Section 2.9, should be determined so that they satisfy appropriate boundary conditions on $S+S_0$ or on S, respectively.

4. Integral Representation Formulas of Electromagnetic Fields (Case of an Open Boundary)

As a continuation of Section 3, we shall study integral representation formulas of electromagnetic fields in a case where a boundary is unclosed, or open.

To begin with, we shall give a definition of an open boundary. Let $L=\sum L_j$, $(j=1,2,\cdots,N)$ be a union of a finite number of piecewise smooth, simple and unclosed line segments of finite length L_j in a plane. If L_j 's are lines of perfect conductor, then we call L a two-dimensional open boundary.

Let $S=\sum S_j$, $(j=1,2,\cdots,N)$ be a union of a finite number of piecewise smooth, simply or multiply connected, orientable, and unclosed surface elements of finite extent S_j. If S_j 's are surfaces of perfect conductor, then we call S a three-dimensional open boundary. (If S_j 's are not of perfect conductor, it is the same as if they do not exist. See the notes which follow the expression (50) below.)

Note that the definition stated here covers a very wide range of open boundaries.

If necessary, the reader should refer to Section 6.18 for explanation of mathematical terminology used here.

4.1 Formulas for a Solution of the Helmholtz Equation.

Let C^* be a closed contour enclosing source points of a field, and $C(R)$ be a circle of radius R enclosing C^* and an open boundary L.

Let L_j be an element of L, and $\tilde{n}_j$ be a normal vector on L_j.

We specify each of the two sides of L_j as its positive or negative side with respect to the direction of $\tilde{n}_j$.

Suppose that L_j^+ and L_j^- are lines parallel to L_j, which exist in the positive and negative sides of L_j, respectively, and that $C_j(\rho)$ are circular arcs of radius ρ with the end points of L_j as their center, so that $C_j = L_j^+ + L_j^- + C_j(\rho)$ composes a closed contour encircling L_j in it, and that C_j converges to L_j in the limit as ρ tends to zero. Let $\tilde{n}_j^+$ and $\tilde{n}_j^-$ be normal vectors on L_j^+ and L_j^-, respectively, which are directed in the inside of C_j. Then, obviously, $\tilde{n}_j = -\tilde{n}_j^+ = \tilde{n}_j^-$. Set $L^+ = \sum L_j^+$, $L^- = \sum L_j^-$, $C(\rho) = \sum C_j(\rho)$ and $C = \sum C_j = L^+ + L^- + C(\rho)$.

[Note] The superscripts $+$ and $-$ have also been used so that $f^+(P)$ and $f^-(P)$ imply the limiting values of a function $f(P')$ in the limit as $P' \notin C$ tends to $P \in C$ from the positive and negative sides of L, respectively.

Assume that there exists a function u^t which satisfies the Helmholtz equation (7) in an unbounded domain which is exterior to L and C^*. Obviously, it satisfies the equation (7) in a domain outside of a fictitious boundary $C = \sum C_j$ as well. Therefore, with the help of (19)$'$, it is represented as

$$u^t(P') = I(P'; C) + I(P'; C^*) + I(P'; C(R))$$

which is rewritten as

$$(\#) \qquad u^t(P') = I(P'; L^+) + I(P'; L^-) + u^{(i)}(P')$$

where we have set

$$u^{(i)}(P') = I(P'; C(\rho)) + I(P'; C^*) + I(P'; C(R)) \tag{43}$$

As was discussed in Section 2.9, $I(P'; C^*)$, $I(P'; C(R))$, $I(P'; C(\rho))$ and therefore an incident field $u^{(i)}$ introduced by (43), are given functions. On the other hand, as can easily be seen, $u^t(P')$ and $u^{(i)}(P')$ are independent of a fictitious boundary C, hence from (#), $I(P'; L^+) + I(P'; L^-)$ is also independent of C. Consequently, (#) holds as it is even if ρ tends to zero, that is, even if $C = L^+ + L^- + C(\rho)$ shrinks to L. In this limit, L^+ and L^- tend to L. Hence, $I(P'; L^+)$ tends to an integral on L whose integrand is the limiting value of that of $I(P'; L^+)$ from the positive side of L. Similarly, $I(P'; L^-)$ tends to an integral on L of the limiting value of the integrand of $I(P';L^-)$ from the negative side of L. Hence, if we refer to (20) and notice the difference of the directions of normals on L^+, L^- and L mentioned above, we can show that

$$I(P';L^+) + I(P';L^-) \to \int_L [\Psi(P',Q)\tau^t(Q) - \{\partial\Psi(P',Q)/\partial n(Q)\}\sigma^t(Q)]ds_Q$$

where we have set

$$\begin{aligned} \tau^t(Q) &= \{\partial u^t(Q)/\partial n(Q)\}^- - \{\partial u^t(Q)/\partial n(Q)\}^+ \\ \sigma^t(Q) &= \{u^t(Q)\}^- - \{u^t(Q)\}^+ \end{aligned} \tag{44}$$

in which $\{\}^+$ and $\{\}^-$ denote, as was noted above, the limiting values of $\{\}$ on L.

Thus we have proved that a solution u^t of the Helmholtz equation (7) in a domain in the exterior of an open boundary L, if exists, is necessarily represented as

$$u^t(P') = \int_L [\Psi(P',Q)\tau^t(Q) - \{\partial\Psi(P',Q)/\partial n(Q)\}\sigma^t(Q)]ds_Q + u^{(i)}(P') \tag{45}$$

where $\sigma^t(Q)$, $\tau^t(Q)$ and $u^{(i)}(P')$ are those defined by (44) and (43), respectively.

Conversely, it is easy to see that $u^t(P')$ defined by (45) in terms of arbitrary integrable functions $\sigma^t(Q)$ and $\tau^t(Q)$ satisfies the Helmholtz equation (7) in the outside of an open boundary L.

We conclude Section 4.1 as follows.

Expression (45), together with (43), furnishes us with a necessary and sufficient formulation of a solution of the Helmholtz equation in a domain exterior to an open boundary L. That is, the solution is given by them, and there is no other solution besides them.

[Note] It is worth noting that the results obtained above for a two dimensional Helmholtz equation hold as they are for a three dimensional Helmholtz equation as well, if we make replacements of a two dimensional open boundary L by a three dimensional open boundary S, the two-dimensional Green function by the three dimensional Green function , and $C(R)$ and $C(\rho)$ in conditions (13) and (15) by $S(R)$ and $S(\rho)$ which will be introduced in Section 4.2 below, etc.

The representation theorem for a three-dimensional problem is stated as follows.

The necessary and sufficient representation of a solution of the three dimensional Helmholtz equation is

$$u^t(P') = \int_S [\Psi(P', Q)\tau^t(Q) - \{\partial\Psi(P', Q)/\partial n(Q)\}\sigma^t(Q)]dS_Q + u(i)(P') \tag{46}$$

where $u^{(i)}(P')$ is a known primary field, and $\Psi(P, Q) = e^{-ikR}/4\pi R$.

4.2 Formulas for Electromagnetic Fields.

Let S^* be a closed surface enclosing sources which generate fields, and $S(R)$ be a sphere of radius R surrounding S^* and an open boundary $S = \sum S_j$.

Let S_j be an element of S, and $\tilde{n}_j$ be a vector normal to S_j. We specify each of the two sides of S_j as its positive and negative side with respect to the direction of $\tilde{n}_j$.

Suppose that S_j^+ and S_j^- are surface elements, lying parallel to S_j in the positive and negative sides of S_j, respectively, that $S_j(\rho)$ are curved circular-cylindrical surfaces of radius ρ whose central lines being peripheries C_j of S_j, and that $F_j = S_j^+ + S_j^- + S_j(\rho)$ composes a closed surface enclosing S_j in it, which converges to S_j when ρ tends to zero.

Let $\tilde{n}_j^+$ and $\tilde{n}_j^-$ be normal vectors on S_j^+ and S_j^-, respectively, which are pointing outward of F_j. Obviously, $\tilde{n}_j = \tilde{n}_j^+ = -\tilde{n}_j^-$.

Set $S^+ = \sum S_j^+$, $S^- = \sum S_j^-$, $S(\rho) = \sum S_j(\rho)$ and $F = \sum F_j$, then $F = S^+ + S^- + S(\rho)$.

Assume that there exist electromagnetic fields $\tilde{E}^t$ and $\tilde{H}^t$ which satisfy the Maxwell equations (4) in an unbounded domain exterior to S and S^*, then obviously, they satisfy (4) in a domain in the outside of a fictitious boundary F as well. Therefore, from (36) and (37), we have

$$\begin{aligned}\tilde{E}^t(P') &= \int_{F+S^*+S(R)} \tilde{G}_E(P', Q; \tilde{J}^t, \tilde{J}^{*t})dS_Q \\ &= \tilde{I}_E(P'; F; \tilde{J}^t, \tilde{J}^{*t}) + \tilde{I}_E(P'; S^*; \tilde{J}^t, \tilde{J}^{*t}) \\ &+ \tilde{I}_E(P'; S(R); \tilde{J}^t, \tilde{J}^{*t}) \\ &= \tilde{E}(P') + \tilde{E}^{(i)}(P')\end{aligned} \tag{47}$$

where we have set

$$\begin{aligned}\tilde{E}(P') &= \tilde{I}_E(P'; S^+; \tilde{J}^t, \tilde{J}^{*t}) + \tilde{I}_E(P'; S^-; \tilde{J}^t, \tilde{J}^{*t}) \\ \tilde{E}^{(i)}(P') &= \tilde{I}_E(P'; S^*; \tilde{J}^t, J^{*t}) \\ &\quad + \tilde{I}_E(P'; S(R); \tilde{J}^t, \tilde{J}^{*t}) + \tilde{I}_E(P'; S(\rho); \tilde{J}^t, \tilde{J}^{*t})\end{aligned} \tag{48}$$

We also have similar expressions for $\tilde{H}^t$, $\tilde{H}$ and $\tilde{H}^{(i)}$, which are obtained from these results by the exchange (23).

As was mentioned in Section 2.9, the first term in the right hand side of (48) is given to express an incident field generated by sources surrounded by S^*, while the second and third terms are given by certain (generalized) radiation condition and edge condition. Therefore, $\tilde{E}^{(i)}$ and $\tilde{H}^{(i)}$ thus obtained above represent known primary incident fields.

Since $\tilde{E}^t(P')$ and $\tilde{E}^{(i)}(P')$ are independent of a fictitious boundaries S^+ and S^-, $\tilde{E}(P')$ remains the same even if S^+ and S^- tend to S. On the other hand, from (32), we have

$$\tilde{I}_E(P'; S^{\pm}; \tilde{J}^t, \tilde{J}^{*t}) = \int_{S^{\pm}} \tilde{G}_E^{\pm}(P', Q; \tilde{J}^t, \tilde{J}^{*t}) dS_Q$$

Since $\tilde{n} = \tilde{n}^+ = -\tilde{n}^-$, it follows that $\tilde{J}^{\pm} = [\tilde{n}^{\pm} \times \tilde{H}^{\pm}] = (-1)^{\pm}[\tilde{n} \times \tilde{H}^{\pm}]$, $\tilde{J}^{*\pm} = (-1)^{\pm}[\tilde{n} \times \tilde{E}^{\pm}]$ and

$$\begin{aligned}\tilde{G}_E^{\pm} = (\pm 1)\{&(1/i\omega\epsilon)([\tilde{n} \times (\tilde{H}^t)^{\pm}] \cdot \nabla)\nabla\Psi \\ &+ [\tilde{n} \times (\tilde{E}^t)^{\pm}] \times \nabla\Psi - i\omega\mu\Psi[\tilde{n} \times (\tilde{H}^t)^{\pm}]\}\end{aligned}$$

As was noted above, superscripts $+$ and $-$ attached to a function indicate limiting values of it on S^+ and S^-, respectively. It is assumed that upper and lower lines go together always.

Consequently, in the limit as S^+ and S^- tend to S, we have

$$\begin{aligned}\tilde{E}(P') &= \tilde{I}_E(P'; S^+; \tilde{J}^t, \tilde{J}^{*t}) + \tilde{I}_E(P'; S^-; \tilde{J}^t, \tilde{J}^{*t}) \\ &= \int_{S^+} \tilde{G}_E^+ dS + \int_{S^-} \tilde{G}_E^- dS \\ &= \int_S \{(1/i\omega\epsilon)(\tilde{J} \cdot \nabla)\nabla\Psi + \tilde{J}^* \times \nabla\Psi - i\omega\mu\Psi\tilde{J}\} dS\end{aligned} \tag{49}$$

where $\tilde{J} = [\tilde{n} \times \{(\tilde{H}^t)^+ - (\tilde{H}^t)^-\}]$ and $\tilde{J}^* = [\tilde{n} \times \{(\tilde{E}^t)^+ - (\tilde{E}^t)^-\}]$. $\tilde{J}$ is called surface electric current density, while $\tilde{J}^*$ is called surface magnetic current density.

We also have, by the exchange (23),

$$\tilde{H}(P') = \int_S \{(i/\omega\mu)(\tilde{J}^* \cdot \nabla)\nabla\Psi + \tilde{J} \times \nabla\Psi + i\omega\epsilon\Psi\tilde{J}^*\}dS \tag{50}$$

If S is not a surface of perfect conductor, then, by virtue of the boundary conditions (8), it follows that $\tilde{J} = 0$ and $\tilde{J}^* = 0$, hence $\tilde{E}(P') = \tilde{H}(P') = 0$, which case is trivial and is out of our consideration. Therefore, we can restrict ourselves to the case where S is a surface of perfect conductor, in which, again by (9), $\tilde{J}^* = 0$.

Therefore,(49) and (50) are reduced to

$$\begin{aligned} \tilde{E}(P') = \int_S \{(1/i\omega\epsilon)(\tilde{J}(Q) \cdot \nabla_Q)\nabla_Q\Psi(P', Q) \\ - i\omega\mu\Psi(P', Q)\tilde{J}(Q)\}dS_Q, \quad (51) \\ \tilde{H}(P') = \int_S \tilde{J}(Q) \times \nabla_Q\Psi(P', Q)dS_Q \end{aligned}$$

where $\tilde{J}(Q) = \tilde{n} \times \{\tilde{H}^+(Q) - \tilde{H}^-(Q)\}$ and $P' \notin S$.

Conversely, it is easy to see that $\tilde{E}$ and $\tilde{H}$ defined by the right members of (51) in terms of an arbitrary integrable density $\tilde{J}$ satisfy the equations (4) everywhere in the outside of S. That is, (51) is a complete representation of a solution of the Maxwell equations as far as a point of observation P' exists in the outside of an open boundary S.

However, we may meet with a trouble when we apply the boundary condition $\tilde{n} \times \tilde{E}^t = 0$ on S, since $\tilde{E}(P')$ is not necessarily finite when P' tends to a point P on S, or on C, because of the singularity of the first term in the integrand of $\tilde{E}(P')$ defined by (51). This fact implies that some additional condition is necessary on $\tilde{J}$ so that it enables (51) to really represent electromagnetic fields.

As will be proved by Theorem A in Section 6.14, surface electric current density $\tilde{J}$ which is induced on an open boundary S must satisfy the condition

$$\tilde{J} \cdot \tilde{P} = 0 \tag{52}$$

on the periphery C of S, where $\tilde{P} = \tilde{n} \times \tilde{t}$, $\tilde{n}$ is a normal vector on S, while $\tilde{t}$ is a tangential vector of C.

Conversely, if $\tilde{E}$ and $\tilde{H}$ are defined by the expressions in the right hand side of (51), in terms of a tangential vector $\tilde{J}$ on S which satisfies

(52) but otherwise arbitrary, then, as will be proved by Theorem B of Section 6.14, expressions (51) are equivalent to

$$\tilde{E}(P') = (-1/i\omega\epsilon) \int_S \{\nabla_Q \cdot \tilde{J}(Q) \nabla_Q \Psi(P', Q) - k^2 \Psi(P', Q) \tilde{J}(Q)\} dS_Q \tag{53}$$

$$\tilde{H}(P') = \int_S \tilde{J}(Q) \times \nabla_Q \Psi(P', Q) dS_Q$$

of which $\tilde{n}(P) \times \tilde{E}(P')$ is known [13] to assume a definite value when $P' \notin S$ tends to $P \in S$. Needless to say, (53) satisfies the Maxwell equations (4) if the condition (52) is taken into consideration.

The reader should refer to Section 6.14 for detailed discussion on the condition (52) and continuity relations between surface electric current and charge densities.

We conclude this section as follows. Expressions (51) and condition (52), or what is the same thing, expressions (53) and (52), furnish us with a necessary and sufficient formulation of electromagnetic fields in a domain bounded by an open boundary. That is, the fields are given by them, and there are no other fields besides them.

5. Analysis of Electromagnetic Scattering by an Open Boundary

In this section, with the help of the integral representation formulas introduced in Sections 4.1 and 4.2, a series expansion approach which solves electromagnetic fields in a domain bounded by an open boundary will be studied. The analysis is practical because concrete procedures of numerical calculation are given; it is general because it is applicable to any geometry of boundary and for any primary incident fields, obtaining fields at everywhere including far and near fields; and it is rigorous because it is confirmed by rigorous mathematical studies.

5.1 Series Expansion Approach to Boundary Value Problems.

As was explained in Section 2.9, in order to determine electromagnetic fields, we must solve some boundary value problems of the Maxwell equations or the Helmholtz equation relating to the boundary of a domain where the fields are considered.

In classical applied mathematics, such problems were solved by the method of separation of variables of a partial differential equation. However, the method was applicable only to limited cases of particular geometry of a closed boundary, and was never applicable efficiently to a case of an open boundary. Therefore, the only effective way is that which is based on integral representation of fields introduced in the previous sections. As was mentioned in Section 2.9, such problems are solved by making the integrals representing secondary fields satisfy boundary conditions.

One approach to make secondary fields meet boundary conditions is to reduce the problem to that of an integral equation derived from an integral representation of fields. In Section 6.10, this approach by an integral equation will be explained in detail. However, as will be shown there, it is not a practical method, especially in a case of an open boundary.

Calderón [3] studied a problem where a boundary is a closed surface S, on which the tangential component of electric field considered in the exterior of S is required to be a prescribed function G. He proved that G can be approximated to any accuracy by an appropriate linear combination of dipole fields whose source points are located in the inside of S, and thus obtained a series expansion of a solution which converges uniformly in a wide sense to the true solution in the domain exterior to S. Yasuura[4] studied his mode expansion technique to solve a two-dimensional field. He introduced a system of wave functions, by which, in a way similar to Calderón, he obtained a series expansion of a solution of the Helmholtz equation in the exterior of a closed contour C. In this case, also, source points or singularities of the wave functions were located in the inside of a closed boundary C. That is, techniques by Calderón and Yasuura depend essentially on the existence of an interior domain where singularities of the expansion functions can be dumped. In other words, their series expansion approach is applicable only to a problem of a closed boundary, and is of no use to a problem where a boundary is open.

In the following, the author's series expansion approach to a problem of an open boundary will be developed. A two-dimensional problem will be studied first, because it is simpler and easier. Then, a three-dimensional problem will be studied along the thread of the study of the two-dimensional problem.

5.2 Fundamental Integral Equation for Two Dimensional Field Scattered by an Open Boundary.

Let L be a two-dimensional open boundary defined in Section 4. L is assumed to be a line of perfect conductor, since otherwise, as is obvious from (44) and (45), L has no effect on fields and appears not to exist.

Let $u^t(P')$ be a two-dimensional E wave, that is, a solution of the Helmholtz equation (7) in the exterior of L. Then, it is given by (45), together with (43) and (44), of Section 4.1.

Because of the boundary condition (10)' prescribed to u^t on a perfect conductor L, σ^t in (44) is zero. On the other hand, by the discussion which derived (22) from (23), τ^t in (44), which is originally the boundary value of u^t, is considered to be the boundary value of the secondary field u itself, and will be denoted as τ. As a consequence, u^t is written, by (45), as

$$u^t(P') = \int_L \Psi(P', Q)\tau(Q)ds_Q + u^{(i)}(P') \tag{54}$$

where $u^{(i)}(P')$ is primary incident field given by (43), in which $I(P'; C(R))$ and $I(P'; C(\rho))$ are known functions given by generalized radiation condition and edge condition, respectively, while $I(P'; C^*)$ is also a given function representing a primary incident field generated by sources located in the inside of C^*.

By the condition $(10)'$ applied on (54), we have

$$\int_L \Psi(P, Q)\tau(Q)ds_Q = g(P) \tag{55}$$

where $P \in C$ and $g(P) = -u^{(i)}(P)$. (55) is the integral equation of Fredholm of the first kind with respect to unknown density τ.

We have thus proved that a two-dimensional field u^t, if exists, is necessarily given by (54) in terms of a solution τ of an integral equation (55).

The converse is also true. That is, if a solution τ of (55) is obtained, and if a function $u^t(P')$ is defined by (54) in terms of τ, then it satisfies all of the Helmholtz equation, the radiation condition and edge condition, and the boundary condition. The proof is given in Section 6.15.

Thus we have proved that (54), together with (55), furnishes the complete formulation of a field corresponding to an arbitrary open boundary and arbitrary primary fields, and that there is no other solution beside it. We call (55) the fundamental integral equation of two-dimensional scattering by an open boundary.

The equation (55) which is an equation of the first kind, is ill posed and is difficult to solve. The author has converted it into a singular integral equation of Cauchy kernel, and studied a successive approximation of a solution. However, by way of saving space, it will not be reproduced here. For the detail, the reader is advised to refer to Hayashi [8].

Because the singular integral equation approach employed in [8] was not practical and could not be generalized so as to solve a three-dimensional problem, the author then switched his study to series expansion approach, the detail of which will be explained in the following section.

5.3 Series Expansion Approach to the Analysis of a Two Dimensional Scattering.

To begin with, we note that a solution of the Helmholtz equation in a domain exterior to an open boundary depends on its boundary value continuously. This means that if the value of a solution of the Helmholtz equation evaluated on a boundary tends to zero, then the solution itself also tends to zero in a domain outside of the boundary. This will be proved if the Green function relating to the boundary is assumed.

Let L be an open boundary, D be the domain outside of L, and P and P' be points such as $P \in L$ and $P' \notin L$, respectively.

Assume that we can find functions $u_n = u_n(P')$, $(n = 1, 2, \cdots)$ such that (i) they satisfy the Helmholtz equation (7) in D, (ii) $u_n(P')$ tend to their boundary values $u_n(P)$ continuously when P' tends to P and vice versa, and (iii) any function $g(P)$ given on L can be approximated by a pertinent linear combination $v_N(P) = \sum_{n=1}^{N} c_n u_n(P)$ with any order of accuracy. Then, since each of $u_n(P')$ satisfies (7), $v_N(P') = \sum_{n=1}^{N} c_n u_n(P')$ also satisfies (7) in D, and assumes a boundary value $v_N(P)$ which is close to $g(P)$. Consequently, by virtue of the continuous dependence of a field on its boundary data mentioned above, $v_N(P')$ is shown to be a such solution of the Helmholtz equation (7) that converges to the solution $u(P')$ whose boundary value being $g(P)$, uniformly in a wide sense in the domain D. This is the fundamental idea of our series expansion approach.

In order to realize this approach, we have to construct a set of functions u_n which satisfies the above mentioned conditions $(i) \sim (\ddot{iii})$. To this end, we need some knowledge from the theory of L_2 space.

A L_2 space on $\mathcal{L}$ is the entirety of square integrable functions defined on $\mathcal{L}$, that is, $L_2 = L_2(\mathcal{L}) = \{f(P); \int_{\mathcal{L}} |f(P)|^2 ds_P < \infty\}$.

An inner product of two elements f and g of $L_2(\mathcal{L})$ is defined by $(f, g) = \int_{\mathcal{L}} f\overline{g}ds$, where $\overline{g}$ denotes a complex conjugate of a complex valued function g. Elements f and g is said to be orthogonal if $(f, g) = 0$. $\|f\| = (f, f)^{1/2} = (\int_{\mathcal{L}} |f(Q)|^2 ds_Q)^{1/2}$ is named the norm of an element f. $\|f - g\| = (\int_{\mathcal{L}} |f(Q) - g(Q)|^2 ds_Q)^{1/2}$ is considered to be the distance between two elements f and g. The limit in a L_2 space $f \to g$ is understood to be $\|f - g\| \to 0$.

A set of elements $\{\varphi_n\}$ is called an orthogonal system if $(\varphi_m, \varphi_n) = \|\varphi_n\|\delta_{m,n}$ hold for all m and n, where δ_{mn} is the Kronecker's delta such as $\delta_{mn} = 1$ if $m = n$, and $= 0$ if $m \neq n$. Furthermore, if $\|\varphi_n\| = 1$ for all n, it is called an orthonormal system.

A system $\{\varphi_n\}$ is said to be complete if an element f of L_2 which is orthogonal to all of $\{\varphi_n\}$ must be zero. Here, we mean by zero an element whose norm is zero, which is denoted by 0.

A set of elements $\{u_n\}$ is said to be dense in L_2 if any element of L_2 can be approximated by a pertinent linear combination of u_n with any order of accuracy. That is, for any element g, we can choose a set of constants $\{c_1, c_2, \cdots, c_N\}$ so that $\|g - \sum c_n u_n\|$ is made as small as we want. An example of a complete system is a system of trigonometric functions $\{(2\pi)^{-1/2}, \pi^{-1/2}\cos nx, \pi^{-1/2}\sin nx\}$ in $L_2(0, 2\pi)$, an expansion by which is a Fourier series expansion.

Now, we can prove the following theorem which is fundamental to our analysis.

<u>*Theorem*</u>

Let $L = \sum_{j=1}^{J} L_j$ be an open boundary, $\{\varphi_{jk}\}$, $k = 1, 2, \cdots$, be a complete system of $L_2(L_j)$, and set

$$u_{jk}(P') = \int_{L_j} \Psi(P', Q)\varphi_{jk}(Q)ds_Q \tag{56}$$

then the entirety U of $u_{jk}(P)$ composes a dense set in $L_2(L)$.

The proof of this theorem is given in Section 6.16.

Note to discriminate $L_2(L_j)$ which denotes a space on a line segment L_j, from $L_2(L)$ which is a space on an open boundary $L = \sum_{j=1}^{J} L_j$, and P' which denotes a point not on L from P on L.

The suffix j of u_{jk} indicates the number of order of an element L_j in L, while the suffix k indicates the number of order of φ_{jk} in the complete system in $L_2(L_j)$. For the sake of simplicity, let us employ a consecutive numbering so that u_{jk} is denoted simply as u_n, and that the aggregate of systems $\{u_{jk}\}$ is denoted simply as $\{u_n\}$.

Then, a set $\{u_n\}$ thus obtained satisfies the conditions $(i) \sim (iii)$ mentioned above. This will be examined in the following.

To begin with, it is noted that, in spite of the singularity of $\Psi(P, Q) = (1/4i)H_0^{(2)}(\kappa\overline{PQ})$ at $P = Q$, a function defined by the integral in the right hand side of (56) varies continuously when $P' \notin L$ tends to, and leaves from, $P \in L$. For the proof, see text books of Potential theory. Next, a function defined by (56)satisfies the Helmholtz equation (7) with respect to P' in a domain D exterior to L, since $\Psi(P', Q)$ satisfies (7) in D. Finally, as the theorem proved just above asserts, the entirety of functions u_{jk} defined by (56), or what is the same thing, the entirety of functions u_n, is dense in $L_2(L)$. Therefore, for any boundary data $g(P)$ given on L, we can choose a set of constants $\{c_n\}$ suitably so that $v_N(P) = \sum_{n=1}^{N} c_n u_n$ approximates $g(P)$ as closely as we want.

A set of constants $\{c_1, c_2, \cdots, c_N\}$ which makes $E = \|g - \sum_{n=1}^{N} c_n u_n\|$ minimum is obtained by solving simultaneous linear equations $\partial E/\partial \overline{c}_n = 0$, or

$$\sum_{m=1}^{N} c_m(u_m, u_n) = (g, u_n), \qquad n = 1, 2, \cdots, N. \tag{57}$$

We call functions u_{jk}, or u_n, defined by (56), "two-dimensional elemental field functions".

We conclude Section 5.3 by summarizing the procedure of series expansion approach to the problem of two-dimensional scattering by an open boundary $L = \sum L_j$ as follows.

Construct a set of two-dimensional elemental field functions u_{jk}, or u_n, by the integral (56) in terms of a complete system $\{\varphi_{jk}\}$ on L_j. Calculate inner products (u_m, u_n) and (g, u_n), where g is the boundary value of a given primary field $u^{(i)}(P')$ mentioned in (54), that is, $g(P) = -u^{(i)}(P)$, and solve the simultaneous linear equations (57). Then,

$$v_N^t(P') = \sum_{n=1}^{N} c_n u_n(P') + u^{(i)}(P') \tag{58}$$

furnishes a solution that converges to the field which corresponds to the boundary value 0, uniformly in a wide sense in a domain outside of L.

5.4 Fundamental Integral Equations for Electromagnetic Fields Scattered by an Open Boundary.

Let $S = \sum_{j=1}^{M} S_j$ be a three-dimensional open boundary defined in Section 4. Every surface element S_j is assumed to be of perfect conductor, since otherwise, as was noted in Section 4.2, S_j makes no effect to the fields and appears not to exist.

As was stated in Section 4.2, electromagnetic fields, generated by known primary fields $E^{(i)}$ and $H^{(i)}$ defined by (48), are given, at $P' \notin S$, by (47);

$$\tilde{E}^t(P') = \tilde{E}(P') + \tilde{E}^{(i)}(P'), \qquad \tilde{H}^t(P') = \tilde{H}(P') + \tilde{H}^{(i)}(P'),$$

where $\tilde{E}(P')$ and $\tilde{H}(P')$ are given by (51);

$$\tilde{E}(P') = \int_S \{(1/i\omega\epsilon)(\tilde{J}(Q)\cdot\nabla_Q)\nabla_Q\Psi(P',Q) - i\omega\mu\Psi(P',Q)\tilde{J}(Q)\}dS_Q$$

$$\tilde{H}(P') = \int_S \tilde{J}(Q)\times\nabla_Q\Psi(P',Q)dS_Q$$

or (53);

$$\tilde{E}(P') = (-1/i\omega\epsilon)\int_S \{(\nabla\cdot\tilde{J}(Q))\nabla_Q\Psi(P',Q) - k^2\Psi(P',Q)\tilde{J}(Q)\}dS_Q$$

$$\tilde{H}(P') = \int_S \tilde{J}(Q)\times\nabla_Q\Psi(P',Q)dS_Q$$

in which $\tilde{J}$ must satisfy (52);

$$\tilde{J}(Q)\cdot\tilde{P}(Q) = 0$$

at any point Q on the periphery C of S.

If (52) is fulfilled, then (51) and (53) are equivalent to each other and satisfy the Maxwell equations (4) in a domain outside of S. On the other hand, because of the boundary condition (9), $\tilde{J}$ must also satisfy an integral equation

$$\tilde{n}(P)\times\int_S \{(1/i\omega\epsilon)(\tilde{J}(Q)\cdot\nabla_Q)\nabla_Q\Psi(P,Q) - i\omega\mu\Psi(P,Q)\tilde{J}(Q)\}dS_Q = \tilde{G}(P) \tag{59}$$

or equivalently

$$\tilde{n}(P)\times(-1/i\omega\epsilon)\int_S \{(\nabla\cdot\tilde{J}(Q))\nabla_Q\Psi(P,Q) - k^2\Psi(P,Q)\tilde{J}(Q)\}dS_Q = \tilde{G}(P) \tag{60}$$

at P on S, where $\tilde{G}(P) = -\tilde{n}(P)\times\tilde{E}^{(i)}(P)$.

Conversely, if $\tilde{J}$ satisfies (52) and either(59) or (60), but otherwise arbitrary, and if $\tilde{E}$ and $\tilde{H}$ are determined by (51) or (53)in terms of the solution $\tilde{J}$, then (47) determines electromagnetic fields, which are generated by given primary fields $\tilde{E}^{(i)}$ and $\tilde{H}^{(i)}$ defined by (48), in a domain exterior to an open boundary S. Thus, (52) and (59), or (52) and (60), constitute the fundamental integral equation of the boundary value problem for an open boundary L.

However, both (59) and (60) are vector valued integral equations of Fredholm of the first kind, and are not solved easily. So, instead, we shall solve the boundary value problem by a series expansion approach which will be studied in the following section.

5.5 Series Expansion Approach to Electromagnetic Fields Scattered by an Open Boundary.

Let us consider a L_2 space on a surface element Σ which is the entirety of square integrable functions on Σ, that is, $L_2(\Sigma) = \{f; \int_\Sigma |f|^2 dS < \infty\}$.

Inner product, norm, orthogonality, dense set, the zero element, etc. are defined in the same way as those defined in the previous section concerning $L_2(\mathcal{L})$ on a line segment $\mathcal{L}$, replacing line integrals on $\mathcal{L}$ by surface integrals on Σ. Especially, a complete system in $L_2(\Sigma)$ is a set of functions φ_n on Σ, such that an element f in $L_2(\Sigma)$ must be zero if $(f, \varphi_n) = 0$ holds for all n.

In a similar way, a vector valued space $\tilde{L}_2(\Sigma)$ is defined as follows.

A tangential vector $\tilde{J}$ on a surface Σ is a vector such as $\tilde{n} \cdot \tilde{J} = 0$, where $\tilde{n}$ is the normal vector on Σ. Any tangential vector $\tilde{J}$ is represented as $\tilde{J} = J_1 \tilde{i}_1 + J_2 \tilde{i}_2$, where $\tilde{i}_1$ and $\tilde{i}_2$ are tangential vectors which are orthogonal to each other and play a role of coordinate vectors on Σ. The space $\tilde{L}_2(\Sigma)$ is the entirety of square integrable tangential vectors on Σ, that is, $\tilde{L}_2(\Sigma) = \{\tilde{J} = J_1 \tilde{i}_1 + J_2 \tilde{i}_2; \int_\Sigma (|J_1|^2 + |J_2|^2) dS < \infty\}$. An inner product of two elements $\tilde{J}$ and $\tilde{K}$ is defined by $(\tilde{J}, \tilde{K}) = \int_\Sigma \{J_1 \overline{K}_1 + J_2 \overline{K}_2\} dS$, and a norm of $\tilde{J}$ by $\|J\| = (\tilde{J}, \tilde{J})^{1/2} = \{\int_\Sigma (|J_1|^2 + |J_2|^2) dS\}$. If $(\tilde{J}, \tilde{K}) = 0$, $\tilde{J}$ and $\tilde{K}$ are said to be orthogonal to each other. An orthonormal system $\{\tilde{J}_n\}$ is such that $(\tilde{J}_m, \tilde{J}_n) = \delta_{m,n}$. An orthonormal set $\{\tilde{J}_n\}$ is called a complete system provided $(\tilde{J}, \tilde{J}_n) = 0$ holds for all n if and only if $\tilde{J} = 0$. A set $\{\tilde{u}_n\}$ in $\tilde{L}_2(\Sigma)$ is called dense if any element $\tilde{G}$ of $\tilde{L}_2$ is approximated by a pertinent linear combination $\sum c_n \tilde{u}_n$ as closely as we want.

Exercise 1

We shall show how a vector valued space $\tilde{L}_2(\Sigma)$ and its complete system $\{\tilde{J}_n\}$ are constructed from a scalar valued space $L_2(\Sigma)$ and its complete system $\{\varphi_n\}$.

Let φ_1 and φ_2 be elements of $L_2(\Sigma)$, and set $\tilde{J} = \varphi_1\tilde{i}_1 + \varphi_2\tilde{i}_2$. Then, obviously, the entirety of $\tilde{J}$ thus obtained coincides with $\tilde{L}_2(\Sigma)$. Let $\{\varphi_n\}$ be a complete system of $L_2(\Sigma)$, and set $\tilde{J}_{2p-1} = \tilde{i}_1\varphi_p$ and $\tilde{J}_{2p} = \tilde{i}_2\varphi_p$, $(p = 1, 2, \cdots)$, then $\{\tilde{J}_n\}$, $(n = 2p-1, 2p, p = 1, 2, \cdots)$ constitutes a complete system in $\tilde{L}_2(\Sigma)$.

Proof

Since $\varphi_1,\ \varphi_2 \in L_2(\Sigma)$, it follows that $\int_\Sigma |\varphi_1|^2 dS < \infty$ and $\int_\Sigma |\varphi_2|^2 dS < \infty$, which imply that $\int_\Sigma\{|\varphi_1|^2 + |\varphi_2|^2\}dS < \infty$, or, $\tilde{J} \in \tilde{L}_2(\Sigma)$. Conversely, if $\tilde{J} \in \tilde{L}_2(\Sigma)$, then $\int_\Sigma\{|\varphi_1|^2 + |\varphi_2|^2\}dS < \infty$, from which we have $\int_\Sigma |\varphi_1|^2 dS < \infty$ and $\int_\Sigma |\varphi_2|^2 dS < \infty$, which imply that $\varphi_1,\ \varphi_2 \in L_2(\Sigma)$. Thus the entirety of $\tilde{J} = \varphi_1\tilde{i}_1 + \varphi_2\tilde{i}_2$, φ_1, $\varphi_2 \in L_2(\Sigma)$, coincides with $\tilde{L}_2(\Sigma)$.

Assume that $\tilde{J} = J_1\tilde{i}_1 + J_2\tilde{i}_2$ satisfies $(\tilde{J},\ \tilde{J}_n) = 0$ for all n, that is, $(\tilde{J},\ \tilde{J}_{2p-1}) = \int_\Sigma J_1\overline{\varphi}_p dS = 0$, and $(\tilde{J},\ \tilde{J}_{2p}) = \int_\Sigma J_2\overline{\varphi}_p dS = 0$ hold for all p, then, because of the completeness of $\{\varphi_p\}$ in $L_2(\Sigma)$, it follows that $J_1 = J_2 = 0$, or $\tilde{J} = 0$. This proves the completeness of $\{\tilde{J}_n\}$ in $\tilde{L}_2(\Sigma)$.

Exercise 2

A note which will be useful in a practical numerical analysis is given here. For example, if S_1 is a circular disk and S_2 is a rectangular disk, then it will be easy to find complete systems in $\tilde{L}_2(S_1)$ and $\tilde{L}_2(S_2)$ separately, but it looks not easy to find a complete system in $\tilde{L}_2(S_1 + S_2)$ on a combined domain $S_1 + S_2$. However, as is shown below, we can construct a complete system on $S = \sum S_k$ with the help of complete systems $\{\tilde{J}_{km}\}$ on S_k.

Though $\tilde{J}_{km}$ is a function defined on S_k, it can be generalized to be defined on $S = \sum S_k$ by setting $\tilde{J}_{km} = 0$ on S_i such as $i \neq k$. Obviously, $\int_S |J_{km}|^2 dS = \int_{S_k} |J_{km}|^2 dS < \infty$, hence $\tilde{J}_{km}$ is not only an element of $\tilde{L}_2(S_k)$ but also an element of $\tilde{L}_2(S)$. By an appropriate alteration of number of order, the aggregate of all elements $\tilde{J}_{km}$, $(k = 1, 2, \cdots, N,\ m = 1, 2, \cdots)$ can be arranged so as to be a set $\{\tilde{J}_n\}$

$(n = 1, 2, \cdots)$ defined on S, whose inner product and norm is obvious. The completeness of $\{\tilde{J}_n\}$ in $\tilde{L}_2(S)$ is proved as follows. Assume that $(\tilde{J}, \tilde{J}_n) = \int_S \tilde{J} \cdot \overline{\tilde{J}_n} dS = 0$ hold for all $\tilde{J}_n = \tilde{J}_{km}$. Since $\int_S \tilde{J} \cdot \overline{\tilde{J}_n} dS = \int_{S_k} \tilde{J} \cdot \overline{\tilde{J}_n} dS = 0$, we have $\tilde{J} = 0$ on S_k because of the completeness of $\tilde{J}_{km}$ in $\tilde{L}_2(S_k)$. Consequently, $(\tilde{J}, \tilde{J}_n) = 0$ for all n imply $\tilde{J} = 0$ on S, proving the completeness of $\{\tilde{J}_n\}$ in $\tilde{L}_2(S)$. Thus we have obtained a complete system in $\tilde{L}_2(S)$.

In terms of a complete system $\{\tilde{J}_n\}$ in $\tilde{L}_2(S)$, we shall define functions $\tilde{E}_n$ and $\tilde{H}_n$ considered in a domain exterior to an open boundary S by

$$\begin{aligned} \tilde{E}_n(P') &= (1/i\omega\epsilon) \int_S \{(\tilde{J}_n(Q) \cdot \nabla_Q) \nabla_Q \Psi(P', Q) + \\ &\quad k^2 \Psi(P', Q) \tilde{J}_n(Q)\} dS_Q \\ \tilde{H}_n(P') &= \int_S \tilde{J}_n(Q) \times \nabla_Q \Psi(P', Q) dS_Q, \qquad (P' \notin S) \end{aligned} \tag{61}$$

and their tangential components $\tilde{u}_n(P)$ on S by

$$\tilde{u}_n(P) = \tilde{n}(P) \times \tilde{E}_n(P), \qquad (P \in S) \tag{62}$$

We call these functions defined by (61) "Elemental electromagnetic field functions", which will play the most important role in the numerical analysis of electromagnetic fields in a domain bounded by an open boundary S.

$\tilde{E}_n$ and $\tilde{H}_n$ defined by (61) are of the same form as $\tilde{E}$ and $\tilde{H}$ introduced by (51) where $\tilde{J}_n$ satisfies the condition (52). Consequently, as was proved in Section 4.2, these functions $\tilde{E}_n$ and $\tilde{H}_n$ satisfy the Maxwell equations and represent the fields in a domain exterior to an open boundary S. Also, the existence of the tangential vectors $\tilde{u}_n$ is shown as follows. Because $\tilde{J}_n$ satisfies (52), by virtue of the Theorem B in Section 6.14, $\tilde{E}_n$ defined by (61) can be shown to be equivalent to

$$\begin{aligned} \tilde{E}_n(P') = (i/\omega\epsilon) \int_S \{\nabla \cdot \tilde{J}_n(Q) \nabla_Q \Psi(P', Q) - \\ k^2 \Psi(P', Q) \tilde{J}_n(Q)\} dS_Q \end{aligned} \tag{63}$$

With respect to $\tilde{E}_n(P')$ defined by (63), it is known[13] that the both limiting values of $\tilde{n}(P) \times \tilde{E}_n(P')$ when $P' \notin S$ tends to $P \in S$ from the positive as well as the negative sides of S exist, and that these limiting values coincide. Consequently, $\tilde{n}(P) \times \tilde{E}_n(P')$, where $\tilde{E}_n(P')$ being defined by (61), exists also in spite of the singularity of the first integrand of (61), hence we can talk about the tangential vectors $\tilde{u}_n$ defined by (62).

Let $\tilde{U}$ be the set of all tangential vectors defined by (62), and $\overline{\tilde{U}}$ be the closure of $\tilde{U}$, the definition of which will be found in Section 6.18. Then, we can prove the following theorem, which makes the basis of our series expansion approach.

Theorem

$\tilde{U}$ is dense in $\tilde{L}_2(S)$. This means that any tangential vector in $\tilde{L}_2(S)$ is approximated by a pertinent linear combination of elements $\tilde{u}_n$ of $\tilde{U}$ as closely as wanted.

The proof of this theorem is given in Section 6.17.

As was mentioned in Section 4.2, a total field $\tilde{E}^t$ is composed of a primary field and the secondary field. While, as was stated in Sections 2.6 and 2.9, a primary field $\tilde{E}^{(i)}$, which is described by (48), is a known field composed of an incident field from given sources and contributions given by generalized radiation and edge conditions. Since S is a surface of perfect conductor, $\tilde{E}^t$ must satisfy the boundary condition (9); $\tilde{n} \times \tilde{E}^t = 0$ on S. Consequently, a secondary field $\tilde{E}$ must satisfy the following boundary condition on S.

$$\tilde{n}(P) \times \tilde{E}(P) = \tilde{G}(P), \qquad \tilde{G}(P) = -\tilde{n}(P) \times \tilde{E}^{(i)}(P) \tag{64}$$

Therefore, the only task left to us in the determination of total fields $\tilde{E}^t$ and $\tilde{H}^t$ is to find secondary fields $\tilde{E}$ and $\tilde{H}$ so that they satisfy the Maxwell equations (4) in the domain bounded by an open boundary S, and that the boundary value $\tilde{n} \times \tilde{E}$ satisfies the condition (64) on S, which problem however, is not easily solved rigorously. Instead, we shall solve it approximately by the series expansion technique which will be developed below.

Let us consider functions $\tilde{E}(P;N)$ and $\tilde{H}(P;N)$ defined by

$$\tilde{E}(P';N) = \sum_{n=1}^{N} c_n \tilde{E}_n(P'), \qquad \tilde{H}(P';N) = \sum_{n=1}^{N} c_n \tilde{H}_n(P') \tag{65}$$

where c's are constants, while $\tilde{E}_n$ and $\tilde{H}_n$ are the elemental electromagnetic field functions defined by (61).

As was noted above, $\tilde{E}_n$ and $\tilde{H}_n$ satisfy the Maxwell equations (4), hence, $\tilde{E}(P;N)$ and $\tilde{H}(P;N)$ also satisfy (4) and make electromagnetic fields in the domain outside of an open boundary S. Furthermore, thanks to the fundamental theorem mentioned above, we are able to choose a set of constants $\{c_n\}$ so that $\|\tilde{G} - \tilde{n} \times \tilde{E}(N)\| = \|\tilde{G} - \tilde{n} \times \sum c_n \tilde{E}_n\| = \|\tilde{G} - \sum c_n \tilde{u}(n)\|$ is made as small as wanted. In other words, we can choose a set of constants $\{c_n\}$ so that the boundary value $\tilde{n} \times \tilde{E}(P;N)$ of the field $\tilde{E}(P;N)$ approximates a given boundary data $\tilde{G}$ of an incident field as precisely as we want. By virtue of the continuous dependence of a field on its boundary data, functions $\tilde{E}(P;N)$ and $\tilde{H}(P;N)$ thus obtained furnish us with electromagnetic fields which converge to the true fields corresponding to the boundary data $\tilde{G}$, uniformly in a wide sense in the domain exterior to an open boundary S.

As is well known, a set of constants $\{c_n\}$ which makes

$$E = \left\| \tilde{G} - \sum c_n \tilde{u}_n \right\|^2 \tag{66}$$

minimum is obtained by solving the following simultaneous linear equations.

$$\sum_{m=1}^{N} c_m (\tilde{u}_m, \tilde{u}_n)_s = (\tilde{G}, \tilde{u}_n)_s, \qquad (n = 1, 2, \cdots, N) \tag{67}$$

where we have set

$$(\tilde{u}_m, \tilde{u}_n)_s = \int_S \tilde{u}_m \cdot \overline{\tilde{u}_n} dS, \quad (\tilde{G}, \tilde{u}_n) = \int_S \tilde{G} \cdot \overline{\tilde{u}_n} dS \tag{68}$$

We conclude Section 5.5 by summarizing the algorithm of our series expansion technique as follows.

Introduce a complete system $\{\tilde{J}_{kn}\}$ in $\tilde{L}_2(S_k)$ such that every element satisfies (52) on the periphery C_k of a surface S_k. Let the union of all sets $\{\tilde{J}_{kn}\}$ be arranged so that it is denoted as $\{\tilde{J}_n\}$. In terms of $\{\tilde{J}_n\}$, construct elemental electromagnetic field functions (61) and their tangential components (62). Calculate inner products $(\tilde{u}_m, \tilde{u}_n)_s$, $(\tilde{G}, \tilde{u}_n)_s$ by (68). Solve equations (67) obtaining solutions $c_1, c_2, \cdots, c_N$. In terms of these values of c, define $\tilde{E}(P;N)$ and $\tilde{H}(P;N)$ by (65). Then (65) gives approximate fields in the domain exterior to an open boundary S, which converge to the true fields in a wide sense in the outside of S as closely as we want.

6. Theorems, Proofs and Comments

In this section, proofs are given to the mathematical results which have appeared in the preceding sections without proof. Furthermore, some theorems and comments which are no less important than the text, are given in Sections 6.10, 11, 14, 18 and 7.

6.1 Integral Representation Formula of a Solution of the Helmholtz Equation, Proof of (18).

Let $\mathcal{L}$ be a closed contour in a plane, D be the interior of $\mathcal{L}$, and $\tilde{n}$ be the normal vector on $\mathcal{L}$ directed outside of D. If a point of observation P exists in D, we exclude it from D by a circle $C(P)$ of a small radius ρ whose center being P, so that $\Psi(P,Q)$ is regular in a domain D' bounded by $\mathcal{L}+C(P)$. Then, with the help of Green's second identity applied to a solution $u^t(P)$ of (7) and $\Psi(P,Q)$ in D', we have

$$\begin{aligned} -\int_{C(P)}[\Psi(P,Q)\{\partial u^t(Q)/\partial n(Q)\}- \\ \{\partial\Psi(P,Q)/\partial n(Q)\}u^t(Q)]ds_Q \\ =\int_{\mathcal{L}}[\Psi(P,Q)\{\partial u^t(Q)/\partial n(Q)\}-\{\partial\Psi(P,Q)/\partial n(Q)\}u^t(Q)]ds_Q \end{aligned} \tag{69}$$

For $Q\in C(P)$, $\overline{PQ}=\rho\ll 1$, hence $\Psi(P,Q)\simeq(-1/2\pi)\ \log\ \rho$ and $\partial\Psi(P,Q)/\partial n(Q)=-\partial\psi(k\rho)/\partial\rho\simeq 1/2\rho$. Consequently, the left member of the expression (69) is reduced to $u^t(P)$ when $\rho\to 0$. If $P\notin D+\mathcal{L}$, we can apply Green's second identity in D without taking C(P) into consideration, hence, the left member of (69) is zero in this case.

Conversely, it is easy to see that the function defined by the right member of (69) satisfies (7), because $\Psi(P,Q)$ satisfies it with respect to P in D. This is the fundamental integral representation theorem of a solution of the Helmholtz equation.

If $\mathcal{L}$ is replaced by $C+C^*+C_0$, these results prove (18).

6.2 Proof; If (13) is Prescribed, $I(P;C(R))$ Vanishes When $R \to \infty$.

Let P be a fixed point in D and $C(R)$ be a circle of radius r with a fixed origin O as its center. Let $OP = \rho$, $OQ = r$, $\theta = \angle POQ$, and $\rho \ll r$, then, $\overline{PQ} = (r^2 + \rho^2 - 2r\rho\cos\theta)^{1/2} \simeq r - \rho\cos\theta$. Hence, we have,

$$\Psi(P,Q) = \{(1-i)/4\}\{\pi\kappa r\}^{-1/2} e^{-i\kappa r + i\kappa\rho\cos\theta},$$
$$\partial\Psi(P,Q)/\partial n(Q) = \{-\kappa(1+i)/4\}\{\pi\kappa r\}^{-1/2} e^{-i\kappa r + i\kappa\rho\cos\theta}.$$

Let $I(P;C(R)) = \int_{C(R)}[\Psi(P,Q)\tau^t(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma^t(Q)]ds_Q$ where $\sigma^t(Q) = u^t(Q)$ and $\tau^t(Q) = \partial u^t(Q)/\partial n(Q)$ on $C(R)$. Consequently, in the limit as $r \to \infty$, we have

$$|I(P;C(R)|^2 \le \int_{C(R)} |(\partial\Psi/\partial R)u^t - \Psi(\partial u^t/\partial n)|^2 ds$$
$$\le const \int_{C(R)} |(\partial u^t/\partial n) + iku^t|^2 ds$$

which tends to zero if the radiation condition (13) is prescribed.

6.3 Proof of (22).

As was proved in Section 6.1, a solution $u^t(P)$ of the Helmholtz equation in a domain D bounded by $C + C^* + C(R)$ is represented by(19)$'$: $u^t(P) = I(P;C) + I(P;C^*) + I(P;C(R))$ where $I(P;\mathcal{L})$ is a function defined by (20); $I(P;\mathcal{L}) = \int_{\mathcal{L}}[\Psi(P,Q)\tau^t(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma^t(Q)]ds_Q$ in which σ^t and τ^t are the boundary values of u^t and $\partial u^t/\partial n$ on $\mathcal{L}$ from the outside of a closed contour $\mathcal{L}$. If we set $u(P) = I(P;C)$, $u^*(P) = I(P;C^*)$ and $u_R(P) = I(P;C(R))$, then $u^t(P) = u(P)+u^*(P)+u_R(P)$, and we also have (#) $\sigma^t = \sigma+\sigma^*+\sigma_R$, $\tau^t = \tau + \tau^* + \tau_R$, the meaning of which may be obvious. On the other hand, $u^*(P)$ which is defined by an integral on C* satisfies the Helmholtz equation everywhere in the exterior of C^*, that is, even in the inside of C. Consequently, by (18) applied to $u^*(P)$ in a domain in the interior of C, we have (§) $\int_C[\Psi(P,Q)\tau^*(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma^*Q)]ds_Q = 0$, where $\sigma^*(Q)$ and $\tau^*(Q)$, respectively, are the boundary values of $u^*(Q)$ and $\partial u^*(Q)/\partial n(Q)$ from the inside of C. However, since u^* and $\partial u^*/\partial n$ are continuous on and in the vicinity of C, these boundary values are the same as those from outside of C, that is, they are the same as those appeared in (#). By the same procedure applied to $u_R(P)$, we have also ($*$) $\int_C[\Psi(P,Q)\tau_R(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma_R(Q)]ds_Q = 0$. Thus we obtained (§), ($*$), (#) and (20); $u(P) = I(P,C) = \int_C[\Psi(P,Q)\tau^t(Q) - \partial\Psi(P,Q)/\partial n(Q)\sigma^t(Q)]ds_Q$, from which we have $u(P) = \int_C[\Psi(P,Q)\tau(Q) - \partial\Psi(P,Q)/\partial n(Q)\sigma(Q)]ds_Q$, where σ and τ are the boundary values of $u(P)$ itself, and not of $u^t(P)$. Thus we have proved (22).

6.4 Proof; (22) Satisfies (7) and (13).

It is obvious that $u(P)$ defined by (22) satisfies (7), since $\Psi(P,Q)$ satisfies it with respect to P when $P \in D$ and $P \neq Q \in C$.

If we set $\overline{OP} = r$, $\overline{OQ} = \rho$ and $\rho \ll r$, then, we have the same approximation concerning $\Psi(P,Q)$ obtained in Section 6.2, which yields $u(P) = I(P;C) = const.r^{-1/2}e^{-ikr}$ and $\partial u/\partial n = \partial u/\partial r = -iku + 0(r^{-3/2})$ when $r = \overline{OP} \to \infty$. Therefore, we have $|\partial u/\partial n + iku|^2 = 0(r^{-3})$, which implies that $u(P)$ defined by (22) satisfies (13).

6.5 Integral Representation Formulas of a Solution of The Maxwell Equations, Proof of (25).

Let D be a domain bounded by a closed surface Σ, and Σ' be an arbitrary closed surface in D, whose inward normal being $\tilde{n}$. Also, let $\tilde{E}$ and $\tilde{H}$ be a solution of the equations (4) in D, and $\tilde{a}$ be an arbitrary constant vector. Set $\tilde{K}_E = (\tilde{n}\cdot\tilde{E})\nabla\Psi + [\tilde{n}\times\tilde{E}]\times\nabla\Psi - i\omega\mu\Psi[\tilde{n}\times\tilde{H}]$ and $\tilde{U}_E = (\tilde{a}\cdot\nabla\Psi)\tilde{E} + \tilde{E}\times[\nabla\Psi\times\tilde{a}] - i\omega\mu\Psi[\tilde{H}\times\tilde{a}]$. Since (4) holds on Σ' and in its interior D', we have $\tilde{a}\cdot\tilde{K}_E = \tilde{n}\cdot\tilde{U}_E$ and $\nabla\cdot\tilde{U}_E = 0$. Hence, when $P \notin D$, with the help of the Gauss theorem, we have $\int_{\Sigma'}\tilde{n}\cdot\tilde{U}_E dS = \int_{D'}\nabla\cdot\tilde{U}_E dV = \tilde{a}\cdot\int_{\Sigma'}\tilde{K}_E dS = 0$, which implies that $\int_{\Sigma'}\tilde{K}_E dS = 0$ because $\tilde{a}$ is arbitrary. Furthermore, since Σ' is arbitrary, we have $\int_\Sigma \tilde{K}_E dS = 0$.

If $P \in D$, we exclude it from D by a sphere $S(P)$ of a small radius ρ with P as its center, so that $\Psi(P,Q)$ is regular in a domain D' bounded by $\Sigma + S(P)$. Then, from what obtained above, we have $-\int_{S(P)}\tilde{K}_E dS = \int_\Sigma \tilde{K}_E dS$. For $Q \in S(P)$, $\overline{PQ} = \rho \ll 1$, $\tilde{n} = \tilde{r}$, and $\nabla\Psi \simeq -\tilde{r}/4\pi\rho^2$. Hence, $\tilde{K}_E \simeq (-1/4\pi\rho^2)\{(\tilde{r}\cdot\tilde{E})\tilde{r} + [\tilde{r}\times\tilde{E}]\times$

$\tilde{r}\} = (-1/4\pi\rho^2)\tilde{E}$, which implies that $-\int_{S(P)} \tilde{K}_E dS \to \tilde{E}(P)$ when $\rho \to 0$. As a consequence, we have $\int_\Sigma \tilde{K}_E dS = \tilde{E}(P)$ when $P \in D$, and $= 0$ when $P \notin D + \Sigma$. Also, by the exchange (23), we have $\int_\Sigma \tilde{K}_H dS = \tilde{H}(P)$ when $P \in D$, and $= 0$ when $P \notin D + \Sigma$.

If Σ is replaced by $C + C^* + C_0$, these results prove (25).

6.6 Proof of (26) and (27).

Assume that $\tilde{E}$ and $\tilde{H}$ satisfy the equations (4) in a domain D bounded by a surface Σ. Set $\tilde{U} = [\tilde{H} \times \nabla\Psi] - i\omega\epsilon\Psi\tilde{E}$, then, by virtue of (4), we have $\tilde{U} = -\nabla \times (\Psi\tilde{H})$ and $\nabla \cdot \tilde{U} = 0$. Hence, $\int_D \nabla \cdot \tilde{U} dV = \int_\Sigma \tilde{n} \cdot \tilde{U} dS = 0$, which, if Σ is replaced by $C + C^* + C_0$, is the first expression of (26). The second expression is obtained from the first one by the exchange (23).

The gradient of (26) is (27), and the divergence of (27) is (26). As a consequence, (26) and (27) are equivalent to each other.

6.7 Proof; If (14) is Prescribed, $\tilde{I}(P; S(R); \tilde{J}^t, \tilde{J}^{*t})$ Vanishes When $R \to \infty$.

At a real boundary where different media are separated, fields may have some discontinuity and therefore the equations (4) may not hold there. Contrary to this, if we consider a surface Σ in a medium where (4) hold, fields are continuous on and in a vicinity of it, and (4) hold on Σ as well. We name such a surface a fictitious boundary.

To begin with, we shall prove the following theorem:

Theorem

If Σ is a fictitious boundary, that is, if (4) holds on Σ as well, then, it holds for $P \notin \Sigma$ that

$$\int_\Sigma \{[\tilde{n} \times \tilde{H}] \cdot \nabla\Psi(P,Q) - i\omega\epsilon(\tilde{n} \cdot \tilde{E})\Psi(P,Q)\} dS_Q = -\int_\Gamma \Psi(P,Q)\tilde{H} \cdot d\tilde{s}_Q \tag{70}$$

$$\int_{\Sigma}\{([\tilde{n}\times\tilde{H}]\cdot\nabla)\nabla\Psi(P,Q)-i\omega\epsilon(\tilde{n}\cdot\tilde{E})\nabla\Psi(P,Q)\}dS_Q = -\int_{\Gamma}\nabla\Psi(P,Q)\tilde{H}\cdot d\tilde{s}_Q \tag{71}$$

where Γ is the periphery of Σ. If Σ is closed, there is no Γ, hence the right members are set to be equal to zero.

Proof of (70) *and* (71).

As was shown in Section 6.6, we have $\tilde{U}=[\tilde{H}\times\nabla\Psi]-i\omega\epsilon\Psi\tilde{E}=-\nabla\times(\Psi\tilde{H})$ which holds even on a fictitious boundary Σ because (4) hold on it. Therefore, with the help of Stokes' theorem, we have $\int_{\Sigma}\tilde{n}\cdot\tilde{U}dS=-\int_{\Sigma}\tilde{n}\cdot\nabla\times(\Psi\tilde{H})dS=-\int_{\Gamma}\Psi\tilde{H}\cdot d\tilde{s}$ which is (70). The gradient of (70) is (71), while the divergence of (71) is (70). Hence, (70) and (71) are equivalent to each other. [Q.E.D]

Now, we shall study $\tilde{I}_E(P:S(R);\tilde{J}^t,\tilde{J}^{*t})=\int_{S(R)}\tilde{G}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q$, where $\tilde{G}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t})$ is the one defined by (29).

Since $S(R)$ is a fictitious closed boundary, the last integral is reduced, with the help of (71), to $=\int_{S(R)}\tilde{K}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q$. Set $\overline{OP}=\rho$, $\overline{OQ}=r$ and $\theta=\angle POQ$ for $Q\in S(R)$, then $\rho\ll r$, and, as was shown in Section 6.2, $R=\overline{PQ}\simeq r-\rho\cos\theta$. Hence, we have, $\Psi=\Psi(P,Q)=A\psi(r)+0(r^{-2})$ where $\psi(r)=e^{-ikr}/4\pi r$ and $A=e^{ik\rho\cos\theta}$. Also, $\nabla\Psi(P,Q)=-ikA\psi(r)\tilde{r}+\tilde{0}(r^{-2})$ where $\tilde{r}=-\tilde{n}=\overrightarrow{OQ}/r$. Hence, $\tilde{K}_E(P,Q)=(\tilde{n}\cdot\tilde{E})\nabla\Psi+[\tilde{n}\times\tilde{E}]\times\nabla\Psi-i\omega\mu\Psi[\tilde{n}\times\tilde{H}]=\{ik\tilde{r}(\tilde{r}\cdot\tilde{E})+ik[\tilde{r}\times\tilde{E}]\times\tilde{r}+i\omega\mu[\tilde{n}\times\tilde{H}]\}A\psi(R)+\tilde{0}(r^{-2})=i\omega\mu\{[\tilde{r}\times\tilde{H}]+(k/\omega\mu)\tilde{E}\}A\psi(r)+\tilde{0}(r^{-2})$. Thus we have $|\tilde{I}_E(P;S(R);\tilde{J}^t,\tilde{J}^{*t})|\le\int_{S(R)}|\tilde{K}_E|dS\le|\omega\mu|\{\int_{S(R)}|A\psi(r)|^2dS\}^{1/2}\cdot\{\int_{S(R)}|[\tilde{r}\times\tilde{H}]+(k/\omega\mu)\tilde{E}|^2dS\}^{1/2}+\tilde{0}(r^{-2})$ which tends to zero when $r\to\infty$, because of the condition (14) and the finiteness of fields proved in Section 6.11. Similar holds for $\tilde{I}_H(P;S(R);\tilde{J}^t,\tilde{J}^{*t})$ as well.

6.8 Proof of (42).

Set $\tilde{E}(P) = \int_S \tilde{G}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q$, $\tilde{E}_{S^*}(P) = \int_{S^*} \tilde{G}_E(P,Q;\tilde{J}^t, \tilde{J}^{*t})dS_Q$ and $\tilde{E}_{S_0}(P) = \int_{S_0} \tilde{G}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q$, where $\tilde{J}^t = [\tilde{n}\times\tilde{H}^t]$ and $\tilde{J}^{*t} = [\tilde{n}\times\tilde{E}^t]$, then (28) is rewritten as $\tilde{E}^t(P) = \tilde{E}(P)+\tilde{E}_{S^*}(P)+\tilde{E}_{S_0}(P)$. Similarly, we have $\tilde{H}^t(P) = \tilde{H}(P)+\tilde{H}_{S^*}(P)+\tilde{H}_{S_0}(P)$, whose meaning is obvious. Hence, it follows that $\tilde{J}^t = [\tilde{n}\times\tilde{H}^t]$ is rewritten as $\tilde{J}^t = \tilde{J} + \tilde{J}_{S^*} + \tilde{J}_{S_0}$ where $\tilde{J} = [\tilde{n}\times\tilde{H}]$, $\tilde{J}_{S^*} = [\tilde{n}\times\tilde{H}_{S^*}]$ and $\tilde{J}_{S_0} = [\tilde{n}\times\tilde{H}_{S_0}]$. Similarly, we have $\tilde{J}^{*t} = \tilde{J}^* + \tilde{J}^*_{S^*} + \tilde{J}^*_{S_0}$, where $\tilde{J}^* = [\tilde{n}\times\tilde{E}]$, $\tilde{J}^*_{S^*} = [\tilde{n}\times\tilde{E}_{S^*}]$ and $\tilde{J}^*_{S_0} = [\tilde{n}\times\tilde{E}_{S_0}]$. Consequently, we have (∗) $\tilde{E}(P) = \int_S \tilde{G}_E(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q = \int_S \tilde{G}_E(P,Q;\tilde{J},\tilde{J}^*)dS_Q + \int_S \tilde{G}_E(P,Q;\tilde{J}_{S^*},\tilde{J}^*_{S^*})dS_Q + \int_S \tilde{G}_E(P,Q;\tilde{J}_{S_0},\tilde{J}^*_{S_0})dS_Q$. While, as is shown by elementary calculation, the functions $\tilde{E}_{S_0}(P) = \int_{S_0} \tilde{G}_E(P, Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q$ and $\tilde{H}_{S_0}(P) = \int_{S_0} \tilde{G}_H(P,Q;\tilde{J}^t,\tilde{J}^{*t})dS_Q$ satisfy (4) independently of $\tilde{J}^t$ and $\tilde{J}^{*t}$ in the inside of S_0 , hence in the inside of a closed surface S as well. Again, with the help of the theorem applied to $\tilde{E}_{S_0}$ and $\tilde{H}_{S_0}$ in the inside of S, it follows that $\int_S \tilde{G}_E(P,Q;\tilde{J}_{S_0},\tilde{J}^*_{S_0})dS_Q = 0$ if P exists in the outside of S , where it should be noted that $\tilde{J}_{S_0} = [\tilde{n}\times\tilde{H}_{S_0}]$ and $\tilde{J}^*_{S_0} = [\tilde{n}\times\tilde{E}_{S_0}]$. Similarly, if we replace S_0 by S^* , we have $\int_S \tilde{G}_E(P,Q;\tilde{J}_{S^*},\tilde{J}^*_{S^*})dS_Q = 0$. Subtracting these two results from (∗), we have $\tilde{E}(P) = \int_S \tilde{G}_E(P,Q;\tilde{J}, \tilde{J}^*)dS_Q$. We also have, by the exchange (23), $\tilde{H}(P) = \int_S \tilde{G}_H(P,Q;\tilde{J}, \tilde{J}^*)dS_Q$. In both these expressions, $\tilde{J}$ and $\tilde{J}^*$ are the boundary values of $\tilde{E}$ and $\tilde{H}$, and not of $\tilde{E}^t$ and $\tilde{H}^t$. Thus we have proved (42).

6.9 (42) Satisfies the Radiation Conditions (14).

In the same way as (28) was reduced to (30) in which $\tilde{A}_P$ and $\tilde{A}^*_P$ were given by (31),(42) is reduced to (30) and (31) if the range of integration $S + S^* + S_0$ is replaced by S and if $\tilde{J}^t$ and $\tilde{J}^{*t}$ are replaced by $\tilde{J}$ and $\tilde{J}^*$, respectively. When $\overline{OP} = r$ and $\overline{OQ} = \rho$ for $Q \in S$ and $P \in S(R)$, then $\rho \ll r$ and $\overline{PQ} \simeq r - \rho\cos\theta$. So, similarly to what stated in Section 6.7, it follows that $\Psi(P,Q) \simeq e^{ik\rho\cos\theta}\psi(r)$, where $\psi(r) = e^{-ikr}/4\pi r$, and hence, $\tilde{A}(P) = \psi(r)\tilde{A}_0$ and $\tilde{A}^*(P) = \psi(r)\tilde{A}^*_0$, where $\tilde{A}_0$ and $\tilde{A}^*_0$ are vectors independent to r . Noting that $\nabla_P\psi(r) = -ik\psi(r)\tilde{r} + \tilde{0}(r^{-2})$, we have $\nabla_P\cdot\tilde{A}^* = -ik\psi(r)\tilde{r}\cdot\tilde{A}_0 + \tilde{0}(r^{-2})$, $\nabla_P\nabla_P\cdot\tilde{A} = -k^2\psi(r)\tilde{r}(\tilde{r}\cdot\tilde{A}_0) + \tilde{0}(r^{-2})$ and

$\nabla_P \times \tilde{A}^* = -ik\psi(r)[\tilde{r}\times\tilde{A}_0^*]+\tilde{0}(r^{-2})$, where $\tilde{r} = \overrightarrow{OP}/r$. On substituting these results in (30), we have $\tilde{E}(P) = i\omega\mu\psi(r)\tilde{r}\times[\tilde{r}\times\tilde{A}_0] - ik\psi(r)\tilde{r}\times\tilde{A}_0^* + \tilde{0}(r^{-2})$. On the other hand, with the help of the exchange (23), it follows that $\tilde{H}(P) = -i\omega\epsilon\psi(r)\tilde{r}\times[\tilde{r}\times\tilde{A}_0^*] - ik\psi(r)\tilde{r}\times\tilde{A}_0 + \tilde{0}(r^{-2})$. Consequently, we have $[\tilde{r}\times\tilde{H}] + (k/\omega\mu)\tilde{E} = \tilde{0}(r^{-2})$, which proves that (42), that is, $\tilde{E}(P) = \tilde{I}_E(P;S;\tilde{J},\tilde{J}^*)$ and $\tilde{H}(P) = \tilde{I}_H(P;S;\tilde{J},\tilde{J}^*)$ satisfy the radiation conditions (14).

6.10 Discussion on Various Integral Equation Approaches to solve Boundary Value Problems for Closed and Open Boundaries.

As was mentioned in 2.9, some boundary value problem must be solved so as to determine electromagnetic fields in a domain bounded by a closed or an open boundary, and various integral equations have been employed for this purpose. Among others, Weyl [1] studied a Dirichlet problem of the Helmholtz equation for a closed boundary, and reduced it to that of solving for an integral equation of Fredholm of the second kind, which is well-posed and well studied problem, and thus established a complete theory of the problem. In this section, Weyl's method, and its vector version to electromagnetic problem as well, will be surveyed. Difference between problems for closed and open boundaries will be discussed, and it will be shown that Weyl's method which was very effective in the case of a closed boundary problem, is of no use in the case of an open boundary problem.

Let C be a closed contour of perfect conductor on xy-pl., D be the unbounded domain in the exterior of C , and $\tilde{n}$ be the normal vector on C directed outward from D . Let u be a two-dimensional E wave, then it must satisfy the Helmholtz equation (7) and the radiation condition (13) in D , and also the Dirichlet boundary condition (§); $u = g$ on C , where $g = -E_z^{(i)}$ is a given incident wave, because of the condition $(10)'$; $E_z + E_z^{(i)} = 0$ on a perfectly conducting boundary C .

Though Weyl studied a three-dimensional Dirichlet problem, we shall consider a E-wave problem, which is a two-dimensional Dirichlet problem. However, it is noted that the two and three-dimensional problems can be treated in quite the same way, only by employing an appropriate Green function in each problem.

As was proved in Section 3.1, a function $u(P)$, which satisfies the Helmholtz equation (7) and the radiation condition (13) in the unbounded domain exterior to C, is necessarily represented by

$$u(P) = \int_C [\Psi(P,Q)\tau(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma(Q)]ds_Q \qquad (72)$$

where $\sigma(Q) = u(Q)$ and $\tau(Q) = \partial u(Q)/\partial n(Q)$. Conversely, it was also proved that a function defined by (72) in terms of arbitrarily chosen σ and τ satisfies (7) and (13) in D.

Before going further, we need the theorem of jump relations, which will be quoted without proof as follows.

Theorem (Jump Relations)

$$\begin{aligned} &\lim \int_C \{\partial\Psi(P',Q)/\partial n(Q)\}\sigma(Q)ds_Q = \\ &\qquad (\pm 1/2)\sigma(P) + \int_C \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma(Q)ds_Q \\ &\lim\{\partial/\partial n(P)\} \int_C \Psi(P',Q)\tau(Q)ds = \\ &\qquad (\mp 1/2)\tau(P) + \int_C \{\partial\Psi(P,Q)/\partial n(P)\}\tau(Q)ds_Q \end{aligned} \qquad (73)$$

where upper and lower lines, respectively, correspond to the limit when P' tends to P from the positive and negative sides of C with respect to the direction of normal $\tilde{n}$ on C.

If the boundary condition (§) is applied to (72), $\sigma(Q)$ in the integrand is replaced by $g(Q)$, and the limit of $u(P')$ as P' tends to $P \in C$ is set equal to $g(P)$. Therefore, with the help of (73), it is shown that

$$\begin{aligned} \lim u(P') = g(P) = (1/2)g(P) + \int_C [\Psi(P,Q)\tau(Q) - \\ \{\partial\Psi(P,Q)/\partial n(Q)\}g(Q)]ds_Q, \end{aligned}$$

that is

$$(\wedge) \qquad \int_C \Psi(P,Q)\tau(Q)ds_Q = (1/2)g(P) + \int_C \{\partial\Psi(P,Q)/\partial n(Q)\}g(Q)ds_Q$$

which is an integral equation of Fredholm of the first kind with respect to the unknown density $\tau(P)$. Conversely, if a solution τ of $(\wedge)$ is found, and if $u(P')$ is defined by (72) in terms of τ, then, the function $u(P')$ thus obtained satisfies the boundary condition (§). In other words, $(\wedge)$ is the fundamental integral equation of the Dirichlet problem composed of (7), (13) and (§), whose solution is given by (72) in terms of a solution τ of $(\wedge)$. That is, the Dirichlet problem has been shown to be equivalent to that of solving an integral equation of Fredholm of the first kind $(\wedge)$. However, as is well known, an integral equation of the first kind is ill posed and is not easily solvable, hence the equation $(\wedge)$ is not a desirable one.

Weyl [1] assumed a solution of the above mentioned problem to be represented by

$$u(P') = \int_C \{\partial\Psi(P', Q)/\partial n(Q)\} k(Q) ds_Q \tag{74}$$

where $k(Q)$ is unknown. Applying the boundary condition and making use of (73), (74) is reduced, in the limit as $P' \notin C$ tends to $P \in C$, to

$$(-1/2)k(P) + \int_C \{\partial\Psi(P, Q)/\partial n(Q)\} k(Q) ds_Q = g(P) \tag{75}$$

Note that (75) is an integral equation of Fredholm of the second kind which is well posed and is well studied.

It is noted that the integral in the right hand side of (74) is of the same form as (72) where $\sigma = -k$ and $\tau = 0$. Since (72) satisfies (7) and (13) independently of the choice of σ and τ, (74) also satisfies (7) and (13), and furthermore, because k satisfies (75), (74) satisfies the boundary condition (§). That is, (74) and (75) furnish us with a complete solution of the problem (7), (13) and (§). The reader is referred to [1] and [2] for the detailed discussion on the existence and uniqueness of a solution of (75), that is, on the Fredholm alternative theorem on it. Now, we shall examine a question how the assumptions $\sigma = -k$ and $\tau = 0$ were possible and what is the physical meaning of k.

As was stated above, a function $u(P')$ defined by (74) satisfies the Helmholtz equation (7), while as was proved in Section 3.1, any solution of (7) is necessarily represented by (72) in terms of boundary values $\sigma(Q)$ and $\tau(Q)$ of its own.

Therefore, setting (74) equal to (72), we see that

$$\int_C [\Psi(P', Q)\tau(Q) - \{\partial\Psi(P', Q)/\partial n(Q)\}\{\sigma(Q) + k(Q)\}] ds_Q = 0 \quad (76)$$

holds identically in the exterior of C, where $\sigma(Q)$ and $\tau(Q)$ are boundary values of the function $u(Q)$ defined by (74) in terms of $k(Q)$ and its derivative $\partial u(Q)/\partial n(Q)$, respectively.

Set

$$v(P') = \int_C [\Psi(P', Q)\hat{\tau}(Q) - \{\partial\Psi(P', Q)/\partial n(Q)\}\hat{\sigma}(Q)] ds \qquad (77)$$

where $\hat{\tau}(Q) = \tau(Q)$ and $\hat{\sigma}(Q) = \sigma(Q) + k(Q)$, then it is obvious that $v(P')$ satisfies (7) in the inside of C, and is identically zero in the outside of C because of (76).

Let $f^+(P)$ and $f^-(P)$, respectively, denote the limiting values of a function $f(P')$ in the limit as $P' \in C$ tends to $P \in C$ from the positive and negative side of C with respect to the direction of normal n on C. Then, with the help of (73) applied to $v(P)$, we have, from (76) and (77), $v^-(P) = 0$ and $[\partial v/\partial n]^- = 0$, and therefore, $v^+(P) - v^-(P) = v^+(P) = -\hat{\sigma}(P)$, and $[\partial v/\partial n]^+(P) - [\partial v/\partial n]^-(P) = [\partial v/\partial n]^+(P) = -\hat{\tau}(P)$.

From these results, we can conclude as follows. The assumptions $\sigma(P) = -k(P)$ and $\tau(P) = 0$ were possible because we assumed the existence of a function $v(P')$ such that it satisfies (7) in the inside of C and is identically zero in the outside of C, and that it takes the limiting values from the inside of C such as $v^+(P) = -\{\sigma(P) + k(P)\}$ and $[\partial v/\partial n]^+(P) = -\tau(P)$, where $\sigma(P)$ and $\tau(P)$ are the limiting values from the outside of C of the function $u(P')$ defined by (74) in terms of $k(P)$. In short, Weyl's assumption was possible because there exists an interior domain of C. that is, because C is closed.

Contrarily to this, if a boundary is open, then there is no interior domain, hence a function such as $v(P')$ introduced by (77) can not find its place to exist and we are unenable to assume a representation of a solution such as (74), and the problem can not be reduced to that of solving for an integral equation of Fredholm of the second kind. Instead, we are compelled to grapple with an integral equation of the first kind, whose solution is not easily obtained. This situation is explained as follows.

Assume that L is an unclosed line segment in a plane, and that $u(P')$ is a solution of the Helmholtz equation (7) in the outside of L, which satisfies the radiation condition (13) and the edge condition (15). Then $u(P')$ is necessarily represented by (45) of Section 4.1 which is of the same form as (22), where however, $\sigma(Q) = u^+(Q) - u^-(Q)$ and $\tau(Q) = [\partial u(Q)/\partial n(Q)]^+ - [\partial u(Q)/\partial n(Q)]^-$. Hence, if boundary conditions are such as $u^+(Q) = u^-(Q) = g(Q)$, (45) is reduced to (#); $u(P') = \int_L \Psi(P', Q)\tau(Q) ds_Q + u^{(i)}(P')$, where $u^{(i)}(P')$ is a known primary field. Again by virtue of the boundary conditions, (#) is reduced, in the limit as $P' \notin L$ tends to $P \in L$, to $\int_L \Psi(P, Q)\tau(Q) ds_Q = g(P) - u^{(i)}(P)$, which is an integral equation of Fredholm of the first kind with respect to τ (Q).

If one wants to avoid an equation of the first kind, he may assume his solution analogously to (74) to be $(*)$ $u(P') = \int_L \{\partial \Psi(P', Q)/\partial n (Q)\} k(Q) ds_Q$, where $k(Q)$ is undetermined yet.

With the help of the jump relations (73) mentioned above, the difference of the limiting values of $u(P')$ defined by $(*)$ in the limit as $P' \notin L$ tends to $P \in L$ from the positive as well as the negative sides of L is shown to be $[u(P)]^+ - [U(P)]^- = k(P)$. On the other hand, by virtue of the boundary condition, the last result proves $k(P) = 0$, which implies that the assumption $(*)$ is impossible, showing that a problem concerning an open boundary is far more difficult than that of a closed boundary.

Next, we shall study a vector version of what was stated above. Since the discussion is similar, the vector version will be explained after the manner of that was described above.

Let S be a closed boundary of perfect conductor, D be the unbounded domain in the exterior of S, and $\tilde{n}$ be the normal on S directed inward of S. As was shown before, electromagnetic fields $\tilde{E}(P)$ and $\tilde{H}(P)$ in D are given by the integral representation (36) or (34), where $\tilde{E}(P)$ should satisfy the boundary condition (9); $\tilde{n}(P) \times \tilde{E}(P) = 0$, or more precisely, $\lim[\tilde{n}(P) \times \tilde{E}(P')] = 0, (P' \to P)$. Because of (9), we set $\tilde{J}^* = [\tilde{n} \times \tilde{E}] = 0$ in the integrand of right members of (34) and (36), and let $P' \notin S$ tend to $P \in S$. Then, the equation $\lim[\tilde{n}(P) \times \tilde{E}(P')] = 0$ leads to an integral equation with respect to $\tilde{J} = [\tilde{n} \times \tilde{H}]$, which however is difficult to solve. Therefore, we shall take a vector version of Weyl's method (74). That is, we assume that $\tilde{E}$ is given by

$$\tilde{E}(P') = \int_S \tilde{k}(Q) \times \nabla_Q \Psi(P', Q) dS_Q + \tilde{E}^{(i)}(P') \tag{78}$$

where $\tilde{E}^{(i)}(P')$ is known primary field composed of integrals on S^* and $S(R)$, and $\tilde{k}(Q)$ is an unknown density.

Before going further, we need the theorem of jump relations, which is a vector version of (73), that is,

Theorem (Jump relations)

In the limit as $P' \notin S$ tends to $P \in S$, we have

$$\begin{aligned} \lim \tilde{n}(P) \times \int_S [\tilde{J}(Q) \times \nabla_Q \Psi(P', Q)] dS_Q = \\ (\pm 1/2)\tilde{J}(P) + \int_S \tilde{n}(P) \times [\tilde{J}(Q) \times \nabla_Q \Psi(P, Q)] dS_Q, \end{aligned} \tag{79}$$

where $+$ or $-$ corresponds to the case in which P' tends to P from the positive or negative side of S with respect to the direction of the normal $\tilde{n}$.

With the help of the relations (79), the condition (9) applied to (78) yields

$$-(1/2)\tilde{k}(P) + \int_S \tilde{n}(P) \times [\tilde{k}(Q) \times \nabla_Q \Psi(P, Q)] dS_Q = \tilde{G}(P), \tag{80}$$

where we have set $\tilde{G}(P) = -\tilde{n}(P) \times \tilde{E}^{(i)}(P)$.

This is an integral equation of Fredholm of the second kind, the analysis of which has been studied well. As was mentioned in Section 3.2, $\tilde{E}$ and $\tilde{H}$ defined by (36) satisfy the Maxwell equations (4) independently of a choice of $\tilde{J}$ and $\tilde{J}^*$. Since (78) is (18) with $\tilde{J} = \tilde{0}$ and $\tilde{J}^* = \tilde{k}$, functions defined by (78) satisfy (4). Furthermore, by virtue of the integral equation (80), they satisfy the boundary conditions (9) as well. Consequently, expression (78) accompanied by (80) furnishes us with a complete set of solutions of the boundary value problem.

It is noted that Calderón [3] was also based on the expression of $\tilde{E}(P')$ defined by (78) above in his multipole expansion theory.

Discussion on the question how the assumption $\tilde{J} = \tilde{0}$ and $\tilde{J}^* = \tilde{k}$ which derived (78) was possible is made in a way similar to that made on the same question concerning (74). Also, when S is open, (79) applied to (78) from the both sides of S yields $\tilde{k} = 0$. That is, the assumption (78) is of no use in the case of an open boundary.

As another example, integral equations which solve a two media problem, which have been introduced in [13], will be described briefly.

Assume that S is a closed surface which separates a bounded interior domain D_1 and an unbounded exterior domain D_2. Let D_m stand for D_1 or D_2 when $m = 1$ or 2, respectively. Let quantities in D_m be discriminated by suffix m. Electromagnetic fields $\tilde{E}_m$ and $\tilde{H}_m$ are given by integrals in (28) if ϵ, μ and $\Psi(P,Q) = e^{-ikR}/4\pi R$ are substituted by those quantities with suffix m in the medium D_m.

Let inward normal on S be $\tilde{n}_m$, and set $\tilde{n} = \tilde{n}_2 = -\tilde{n}_1$, then, $\tilde{n}_m = (-1)^m \tilde{n}$ where $\tilde{n}$ is the normal vector directed from D_1 to D_2. By virtue of the boundary conditions (8), we have $\tilde{n} \times (\tilde{H}_2 - \tilde{H}_1) = 0$, or $\tilde{J}_2 = -\tilde{J}_1$, which we set $= \tilde{J}$. Similarly, from $\tilde{n} \times (\tilde{E}_2 - \tilde{E}_1) = 0$, we have $\tilde{J}_2^* = -\tilde{J}_1^*$, which we set $= \tilde{J}^*$. That is, we have $\tilde{J}_m = (-1)^m \tilde{J}$ and $\tilde{J}_m^* = (-1)^m \tilde{J}^*$.

Let the representation (28) of $\tilde{H}_m(P_m)$ evaluated at a point P_m in D_m be multiplied by $\mu_m \tilde{n}(P)$, and let $\tilde{J}_m$ and $\tilde{J}_m^*$ in the representation be replaced by $(-1)^m \tilde{J}$ and by $(-1)^m \tilde{J}^*$, respectively. Then, $\mu_2 \tilde{n}_2(P) \times \tilde{H}_2(P_2) - \mu_1 \tilde{n}_1(P) \times \tilde{H}_1(P_1)$, where $P \in S$, $P_1 \in D_1$ and $P_2 \in D_2$, is represented by certain integrals composed of the integrals in the right hand sides of (28). If we let $P_1 \to P$ and $P_2 \to P$, then the last quantity tends to $\mu_2 \tilde{n}_2(P) \times \tilde{H}_2(P) - \mu_1 \tilde{n}_1(P) \times \tilde{H}_1(P) = \mu_2 \tilde{J}_2(P) - \mu_1 \tilde{J}_1(P) = (\mu_1 + \mu_2)\tilde{J}(P)$. While, the limiting value of the corresponding integral is calculated with the help of the jump relations (79). Then, equating these two limiting values, we have

$$\begin{aligned}\tilde{J}(P) + \{2/(\mu_1+\mu_2)\} \int_S &[(i/\omega)(\tilde{J}^*(Q)\cdot\nabla_Q)\tilde{n}(P) \times \nabla_Q\{\Psi_1(P,Q)- \\ &\Psi_2(P,Q)\} + \tilde{n}(P) \times [\tilde{J}(Q) \times \nabla_Q\{\mu_1\Psi_1(P,Q)- \\ &\mu_2\Psi_2(P,Q)\}] + i\omega\tilde{n}(p) \times \tilde{J}^*(Q)\{\mu_1\epsilon_1\Psi_1(P,Q)- \\ &\mu_2\epsilon_2\Psi_2(P,Q)\}]dS \\ = \{2/(\mu_1+\mu_2)\}\tilde{n}(P) \times &\{\mu_1 H_1^{(i)}(P) + \mu_2 \tilde{H}_2^{(i)}(P)\} \qquad (81)\end{aligned}$$

and the equation obtained from (81) by the exchange (23). These equations constitute a system of simultaneous integral equations of Fredholm of the second kind, which solves the two media problem mentioned above. That is, $\tilde{E}(P)$ and $\tilde{H}(P)$, defined by (28) in terms of solutions $\tilde{J}$ and $\tilde{J}^*$ of the equations (81), satisfy the Maxwell equations (4) and the boundary conditions (8). It is noted that, in spite of the singularity of $\Psi(P,Q)$, the kernels of these equations are square integrable, which make the equations (81) well posed.

The reader is referred to [13] for the detailed study on these equations.

6.11 Theorems on and Related to the Uniqueness of Fields.

Theorems on the uniqueness of fields are important not only because of their mathematical interest but also by their role in our series expansion approach to an open boundary problem.

Wilcox proved his uniqueness theorem, expansion theorem and finite theorem, of a solution of the three-dimensional Helmholtz equation [14], and of electromagnetic fields [17], both in a domain bounded by a closed boundary. In this section, following and modifying Wilcox's works on closed boundary problems, we shall prove the corresponding theorems for a two-dimensional E wave, and also three-dimensional electromagnetic fields, in a domain bounded by an open boundary.

As was introduced in Section 3.2, let S be a union of closed surfaces, $S(R)$ be a sphere of radius R enclosing S, and $\tilde{E}(P)$ and $\tilde{H}(P)$ be electromagnetic fields in the domain exterior to S, which satisfy the radiation conditions (14). Then we have

$$\lim_{R\to\infty}[\int_{S(R)}\{|\tilde{r}\times\tilde{E}|^2+|(k/\omega\epsilon)\tilde{H}|^2\}dS-$$
$$2(\mathrm{Im}k)\int_{D(R)}\{|(\omega\mu/k)\tilde{H}|^2+|\tilde{E}|^2\}dV]$$
$$=2\mathrm{Re}\{(\overline{k/\omega\epsilon})\int_S[\tilde{n}\times\tilde{E}]\cdot\overline{\tilde{H}}dS\}$$

$$\lim_{R\to\infty}[\int_{S(R)}\{|\tilde{r}\times\tilde{H}|^2+|(k/\omega\mu)\tilde{E}|^2\}dS- \\ 2(\mathrm{Im}k)\int_{D(R)}\{|(\omega\epsilon/k)\tilde{E}|^2+|\tilde{H}|^2\}dV] \\ =2\mathrm{Re}\{(\overline{-k/\omega\mu})\int_S[\tilde{n}\times\tilde{H}]\cdot\overline{\tilde{E}}dS\} \tag{82}$$

where $\tilde{r}$ is the radial vector of $S(R)$, and $D(R)$ is the domain surrounded by S and $S(R)$.

Proof

Suppose that S' is a closed surface enclosing S in the interior of $D(R)$, and that D' is the domain bounded by $\Sigma = S' + S(R)$. Since (4) holds on and in the vicinity of Σ, we have, with the help of the Gauss theorem, $i\overline{\omega\mu}\int_\Sigma[\tilde{n}\times\tilde{E}]\cdot\overline{\tilde{H}}dS = \int_\Sigma[\tilde{n}\times\tilde{E}]\cdot\overline{\nabla\times\tilde{E}}dS = -\int_{D'}\nabla\cdot[\tilde{E}\times\overline{\nabla\times\tilde{E}}]dV = -\int_{D'}\{|\omega\mu\tilde{H}|^2-\overline{k^2}|\tilde{E}|^2\}dV$. Let S' tend to S, note that $\tilde{n}=-\tilde{r}$ on $S(R)$, multiply the last expressions by $(\overline{i/k})$, and take the twice of the real part of the result, then we have

$$(*)\qquad 2\mathrm{Re}(\overline{k/\omega\epsilon})\int_{S(R)}[\tilde{r}\times\tilde{E}]\cdot\overline{\tilde{H}}dS = 2\mathrm{Re}(\overline{k/\omega\epsilon})\int_S[\tilde{n}\times\tilde{E}]\cdot\overline{\tilde{H}}dS+ \\ 2(\mathrm{Im}k)\int_{D(R)}\{|(\omega\mu/k)\tilde{H}|^2+|\tilde{E}|^2\}dV$$

Similar result is obtained as a counterpart of $(*)$ by the exchange (23).

On the other hand, from the radiation conditions (14), we have

$$\int_{S(R)}|[\tilde{r}\times\tilde{E}]-(k/\omega\epsilon)\tilde{H}|^2dS = \int_{S(R)}\{|[\tilde{r}\times\tilde{E}]|^2+|(k/\omega\epsilon)\tilde{H}|^2\}dS- \\ 2\mathrm{Re}(\overline{k/\omega\epsilon})\int_{S(R)}[\tilde{r}\times\tilde{E}]\cdot\overline{\tilde{H}}dS = \tilde{0}(R^{-1}),$$

and its counterpart obtained by (23). Substituting the last result in $(*)$, we have (82). [Q.E.D]

We shall introduce a series expansion theorem for the fields $\tilde{E}(P)$ and $\tilde{H}(P)$ which satisfy the conditions mentioned above, Set $r = \overline{OP}$, $\rho = \overline{OQ}$ and $c = max.\rho$ for $Q \in S$. Then, at a point P such as $2.5c < r$, we have

$$\tilde{E}(P) = (e^{-ikr}/4\pi r) \sum_{m=0}^{\infty} \tilde{E}_m(\theta, \varphi) r^{-m} \tag{83}$$

where θ and φ are spherical coordinates of P. $\tilde{E}_m$ is determined by the recursion formula

$$-2ikm\tilde{E}_m = m(m-1)\tilde{E}_{m-1} + \mathcal{D}\tilde{E}_{m-1}, \qquad (m = 1, 2, \cdots), \tag{84}$$

where $\mathcal{D}$ is the Beltrami operator which is

$$\mathcal{D} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

Corresponding to (83) and (84), we also have

$$\tilde{H}(P) = (e^{-ikr}/4\pi r) \sum_{m=0}^{\infty} \tilde{H}_m(\theta, \varphi) r^{-m} \tag{83$'$}$$

$$-2ikm\tilde{H}_m = m(m-1)\tilde{H}_{m-1} + \mathcal{D}\tilde{H}_{m-1}, \qquad (m = 1, 2, \cdots) \tag{84$'$}$$

Note

When $r \gg 1$, (83) is reduced to $\tilde{E}(P) \simeq (e^{-ikr}/4\pi r)\tilde{E}_0$. That is, $\tilde{E}_0$ is the radiation pattern of field $\tilde{E}(P)$. Similar holds for $\tilde{H}(P)$ from (83)$'$.

Proof

Set $R = \overline{PQ} = r\sqrt{1 - X}$, where $X = x(2\rho cos\theta - \rho^2 x)$, $x = 1/r$ and $\theta = \angle POQ$. Also set $\Psi(P, Q) = \psi(r)\Phi(P, Q)$, where $\psi(r) = e^{ikr}/4\pi r$ and $\Phi(P, Q) = (1 - X)^{-1/2} exp.\{-ikr(\sqrt{1 - X} - 1)\}$. It is easy to see that $|X| < 1$ holds if $2.5c < r$. Therefore, $\sqrt{1 - X}$, and $\Phi(P, Q)$ as well, are analytic with respect to x, hence we can expand as $\Phi(P, Q) = \sum r^{-n}\Phi_n(P, Q)$ and $\nabla_Q\Phi(P, Q) = \sum r^{-n}\nabla_Q\Phi_n(P, Q)$. Note that these series are termwise differentiable and integrable. On the other hand, as has been proved before, $\tilde{E}(P)$ which satisfies the Maxwell equations (4) and the radiation condition (14), is represented as $\tilde{E}(P) = \int_S \tilde{G}_E(P,Q;\tilde{J}\ ,\ \tilde{J}^*)dS_Q$, which is then reduced, by virtue of the results obtained above, to (83);

$$\tilde{E}(P) = \psi(r)\sum r^{-n}\int_S[(i\omega\epsilon)^{-1}(\tilde{J}\cdot\nabla)\nabla\Phi_n + \tilde{J}^*\times\nabla\Phi_n - i\omega\mu\Phi_n\tilde{J}]dS = \psi(r)\sum r^{-n}\tilde{E}_n(\theta,\varphi)$$

Furthermore, if (83) is substituted, we see that $\triangle\tilde{E}+k^2\tilde{E} = (\partial^2/\partial r^2 + 2r^{-1}\partial/\partial r + r^{-2}\mathcal{D} + k^2)\tilde{E} = 0$ is reduced to $\sum e^{-ikr}r^{-(m+2)}$ $\{2ikm\tilde{E}_m + m(m-1)\tilde{E}_{m-1} + \tilde{E}_{m-1}\} = 0$, which leads to (84).

Now, we shall apply these results so as to obtain similar results in a case of an open boundary. As was introduced in Section 4.2, let S be a union of unclosed surface elements, $S(R)$ be a sphere of radius R enclosing S, and $\tilde{E}(P)$ and $\tilde{H}(P)$ be electromagnetic fields in a domain exterior to S, which satisfy the radiation conditions (14) and the edge condition (16). Then we have

$$\lim_{R\to\infty}[\int_{S(R)}\{|\tilde{r}\times\tilde{E}|^2 + |(k/\omega\epsilon)\tilde{H}|^2\}dS - 2(\mathrm{Im}k)\int_{D(R)}\{|(\omega\mu/k)\tilde{H}|^2 + |\tilde{E}|^2\}dV] = 2\mathrm{Re}\{(\overline{k/\omega\epsilon})\int_S[\tilde{n}\times\tilde{E}](\overline{\tilde{H}}^+ - \overline{\tilde{H}}^-)dS\}$$

$$\lim_{R\to\infty}[\int_{S(R)}\{|\tilde{r}\times\tilde{H}|^2 + |(k/\omega\mu)\tilde{E}|^2\}dS - 2(\mathrm{Im}k)\int_{D(R)}\{|(\omega\epsilon/k)\tilde{E}|^2 + |\tilde{H}|^2\}dV] = 2\mathrm{Re}\{(\overline{-k/\omega\mu})\int_S[\tilde{n}\times\overline{\tilde{E}}]\cdot(\tilde{H}^- - \tilde{H}^+)dS\} \quad (85)$$

On the other hand, the series expansion theorem (83) holds as it is even if S is unclosed.

Proof

As was introduced in Section 4.2, let $S^+ + S^- + S(\rho)$ be a closed surface enclosing the open surface S. Then the expressions in (82) hold if the integral in the right member of them are replaced by the sum of integrals on S^+, S^- and $S(\rho)$. If one notes that the normal $\tilde{n}$ on S^+ is the same as $\tilde{n}$ on S, but $\tilde{n}$ on S^- is -1 times of $\tilde{n}$ on S, then, by virtue of the boundary condition $\tilde{n} \times \tilde{E}^+ = \tilde{n} \times \tilde{E}^-$, the sum of integrals on S^+ and S^- tends to the right members of (85). On the other hand, integrals on $S(\rho)$ vanish in the limit as $\rho \to 0$, by virtue of the edge condition. Thus, we have (85).

(83) holds even if S is open, because the proof of (83) stated above is easily seen to be valid if S is understood to be a union of unclosed surfaces.

Similar results hold in the two-dimensional case as well.

Let C and $C(R)$ be those introduced in Section 3.1, and $u(P)$ satisfy the Helmholtz equation (7) and the radiation condition (13) in a domain exterior to C. If C is unclosed, then the edge condition (15) is supplemented. Then, corresponding to (82) and(85), we have the following theorems. That is, we have

$$\lim_{R\to\infty}[\int_{C(R)}\{|\partial u/\partial n|^2+|ku|^2\}ds- 2(\mathrm{Im}k)\int_{S(R)}\{|\nabla u|^2+|ku|^2\}dS] = -2\mathrm{Im}\{k\int_C u(\overline{\partial u/\partial n})ds\} \tag{86}$$

when C is closed, and

$$\lim_{R\to\infty}[\int_{C(R)}\{|\partial u/\partial n|^2+|ku|^2\}ds - 2(\mathrm{Im}k)\int_{S(R)}\{|\nabla u|^2+|ku|^2\}dS] = -2\mathrm{Im}\{k\int_C u[(\overline{\partial u/\partial n})^+ - (\overline{\partial u/\partial n})^-]ds\} \tag{87}$$

when C is unclosed, where $S(R)$ is the domain enclosed by C and $C(R)$.

The procedure for the proof of (86) and (87) may be guessed if one refers to the proof of (82) and (85), hence it will not be described here.

As a series expansion theorem corresponding to (83), we have

$$u(P) = r^{-1/2}e^{-ikr}\sum C_m(\theta)r^{-m} \tag{88}$$

where (r,θ) is the polar coordinate of P. The coefficients C_m are determined by the following recursion formula.

$$2ikmC_m(\theta) = -(m-1/2)^2C_{m-1}(\theta) - \mathcal{D}C_{m-1}(\theta) \tag{89}$$

The proof of (88) is as follows. As it can be seen from (19), a solution of (7) which satisfies (13) in a domain outside of C is given by $(\#) u(P) = \int_C [\Psi(P,Q)\tau(Q) - \{\partial\Psi(P,Q)/\partial n(Q)\}\sigma(Q)] dS_Q$. If the expansion of the Hankel function $\Psi(P,Q) = (1/4i)H_o^{(2)}(kR) = (1/4i)\Sigma e^{im(\theta-\varphi)} H_m^{(2)}(kr)J_m(k\rho)$ is substituted and if we set $u_m = (1/4i)\int_C e^{im\varphi}[J_m(k\rho)\tau(Q) - kJ_m(k\rho)\sigma(Q)]dS_Q$, where (ρ,φ) is the coordinate of Q , $(\#)$ is reduced to $u(P) = \sum u_m e^{im\theta} H_m^{(2)}(kr)$. If $kr \gg 1, H_m^{(2)}(kr)$ is replaced by $= r^{-1/2}e^{-ikr}\sum a_m r^{-m}$ where a_m are constants. Thus, we have (88). On substituting (88) in $\triangle u + k^2 u = (\partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\varphi^2 + k^2)u = 0$, we have (89).

Now, we shall prove the theorems of uniqueness and finiteness of fields, which are stated as follows.

Theorem

An electromagnetic field $\{\tilde{E}, \tilde{H}\}$ in an unbounded domain in the exterior of S , which satisfies the Maxwell equations (4), the radiation conditions (14), and also the edge condition (16) if S has sharp edges, is identically zero if it assumes the boundary value $[\tilde{n} \times \tilde{E}] = 0$ on S . This holds in all cases where S is closed or unclosed, and where $\mathrm{Im}k < 0$ or $\mathrm{Im}k = 0$.

Proof

As has been proved above, if $\tilde{E}$ and $\tilde{H}$ satisfy (4), (14) and (16), (82), and (83) hold when S is closed, while (85) and (83) hold when S is unclosed. Therefore, if $[\tilde{n} \times \tilde{E}] = 0$ on S , the right members of (82) and (85) are zero, hence the left members of them are zero as well. On the other hand, because $\mathrm{Im}k \leq 0$, each term in the left hand sides of (82) and (85) is non negative. As a consequence, each term must be zero individually.

If $\mathrm{Im}k < 0$, from the last term in the left hand side of (82) or(85), we have $\lim \int_{S(R)} |\tilde{E}|^2 dV = 0$ and $\lim \int_{D(R)} |\tilde{H}|^2 dV = 0$, which imply that $\tilde{E}$ and $\tilde{H}$ are identically zero in the outside of S .

If $\mathrm{Im}k = 0$, from the first term of (82) or (85), we have $\lim \int_{S(R)} |\tilde{E}|^2 dS = 0$. While, with the help of (83) which holds for both closed and unclosed S , the last result is reduced to $\int_{S(R)} |\tilde{E}|^2 dS = (1/4\pi)^2 \int |\tilde{E}_0|^2 d\Omega + 0(R^{-1})$, where $d\Omega$ is a solid angle sustained by dS of $S(R)$.

Consequently, we have $\tilde{E}_0 = 0$. With the help of the recursion formula (84), we also have $\tilde{E}_m = 0$ for all m. Hence, by virtue of (83), we have $\tilde{E} = 0$. Similarly, we can prove $\tilde{H} = 0$ if we use $(83)'$ and $(84)'$ instead of (83) and (84).

Corollary 1 (*Finiteness Theorem*)

$$\int_{S(R)} |E|^2 dS < \infty, \qquad \int_{S(R)} |H|^2 dS < \infty. \tag{90}$$

and furthermore, when $\mathrm{Im} k < 0$,

$$\int_{D(R)} |E|^2 dV < \infty, \qquad \int_{D(R)} |H|^2 dV < \infty. \tag{91}$$

hold for any R.

Proof

Since the right member of (82), and of (85) as well, is finite independently of R, (90) and (91) are obvious.

Corollary 2

For any choice of generalized radiation condition and generalized edge condition, the corresponding fields are unique.

Proof

As was mentioned in Section 2.9, a total field $\tilde{E}^t$ is represented by a sum of a primary field $\tilde{I}_E(S^*)+\tilde{I}_E(S(R))+\tilde{I}_E(S(\rho))$ and a secondary field $\tilde{I}_E(S)$, where a primary field must be given by a generalized radiation condition and a generalized edge condition since otherwise we have no way to know what it is. Assume that there are two total fields $\tilde{E}^{t'}$ and $\tilde{E}^{t''}$ corresponding to the same primary field such as $\tilde{E}^{t'} = \tilde{I}'_E(S) + \tilde{I}_E(S^*) + \tilde{I}_E(S(R)) + \tilde{I}_E(S(\rho))$ and $\tilde{E}^{t''} = \tilde{I}''_E(S) + \tilde{I}_E(S*)+\tilde{I}_E(S(R))+\tilde{I}_E(S(\rho))$, then obviously we have $\tilde{E}^t = \tilde{I}_E$ where $\tilde{E}^t = \tilde{E}^{t'} - \tilde{E}^{t''}$ and $\tilde{I}_E = \tilde{I}'_E - \tilde{I}''_E$. Suppose that $\tilde{E}^{t'}$ and $\tilde{E}^{t''}$ satisfy same boundary condition on S, then we have $\tilde{n} \times \tilde{E}^t = \tilde{n} \times \tilde{I}_E = 0$, which implies that, by virtue of the Theorem mentioned above, $\tilde{I}_E = 0$, or $\tilde{I}'_E = \tilde{I}''_E$. That is, there is a unique field corresponding to each of a given generalized radiation and edge conditions.

Similar theorems of uniqueness and finiteness hold also for a two-dimensional field.

If $u = 0$ on C, then the right member of (86) when C is closed, or of (87) when C is unclosed, is zero, hence their left member is zero as well. Because $Imk \leq 0$, this implies that each term must be zero individually. Consequently, when $Imk < 0$, the last term of the left member shows that $\lim_{R\to\infty} \int_{S(R)} |u|^2 dS = 0$, which implies $u = 0$. If $Imk = 0$, we have $\lim \int_{C(R)} |u|^2 dS = 0$, which is reduced, with the help of the expansion theorem (88), to $\lim \int_0^{2\pi} [C_o|^2 d\theta = 0$. Therefore, we have $C_o = 0$ which then proves $C_n = 0$ for all n with help of (89). Hence, we have $u = 0$.

Because the right member of (86) and (87) are finite independently of R, it is easy to see that we have the following theorem of finiteness. That is, for any R, it holds that

$$\int_{C(R)} |u|^2 dS < \infty \qquad \int_{C(R)} |\partial u/\partial n|^2 dS < \infty \tag{92}$$

Furthermore, if $Imk = 0$,

$$\int_{S(R)} |u|^2 dS < \infty \qquad \int_{S(R)} |\nabla u|^2 dS < \infty \tag{93}$$

6.12 Proof; If (15) is Prescribed, $I(P';C(\rho))$ Vanishes When $\rho \to 0$.

$I(P';C(\rho))$ is an integral defined by (20) with $\mathcal{L}$ replaced by $C(\rho)$, where $C(\rho)$ denotes circular arcs of radius ρ around end points of an open boundary L.(See Section 4.1)

It is noted that, for a point $P' \notin L$, there exists a constant ρ_0 such that $\overline{P'Q} \neq 0$ holds for $Q \in C(\rho)$ if $\rho < \rho_0$. Hence, the absolute values of $\Psi(P', Q) = (1/4i)H_0^{(2)}(k\overline{P'Q})$ and its derivatives are bounded by some constant M. Therefore, it follows that $|I(P';C(\rho))| < \int_{C(\rho)} \{|\Psi||\tau| + |\partial\Psi/\partial n||\sigma|\} ds < M \int_{C(\rho)} \{|\partial u/\partial n| + |u|\} ds$ which tends to zero if (15) is prescribed.

6.13 Proof; If (16) is Prescribed, $\tilde{I}(P'; S(\rho); \tilde{J}, \tilde{J}^*)$ Vanishes When $\rho \to 0$.

$\tilde{I}_E(P'; S(\rho); \tilde{J}, \tilde{J}^*)$ and $\tilde{I}_H(P'; S(\rho); \tilde{J}, \tilde{J}^*)$ are integrals defined by (32) with Σ replaced by $S(\rho)$, where $S(\rho)$ denotes curved circular-cylindrical tube of radius ρ defined in Section 4.2.

It is noted that $(\tilde{J} \cdot \nabla)\nabla\Psi = ([\tilde{n} \times \tilde{H}] \cdot \nabla)\nabla\Psi$ is a sum of products of a component of $\tilde{H}$ and a derivative of Ψ. Hence, by the boundedness of Ψ and its derivative at $P' \notin S$ noticed in Section 6.12, an elementary calculation shows that $|(\tilde{J} \cdot \nabla)\nabla\Psi| < M|\tilde{n} \times \tilde{H}|$, where M is a constant. Similarly, we have $|\tilde{J}^* \times \nabla\Psi| < M|\tilde{n} \times \tilde{E}|$ and $|-i\omega\mu\tilde{J}| < M|\tilde{n} \times \tilde{H}|$. So we have $|\tilde{I}_E(P'; S(\rho); \tilde{J}, \tilde{J}^*) < \int_{S(\rho)} |(1/i\omega\epsilon)(\tilde{J} \cdot \nabla)\nabla\Psi + \tilde{J}^* \times \nabla\Psi - i\omega\mu\Psi\tilde{J}|dS < M \int_{S(\rho)} \{|\tilde{E}| + |\tilde{H}|\}dS$. In a way similar to this, we also have $|\tilde{I}_H(P'; S(\rho); \tilde{J}, \tilde{J}^*)| < M \int_{s(\rho)} \{|\tilde{E}| + |\tilde{H}|\}dS$. Therefore, if (16) is prescribed, both of these integrals vanish when $\rho \to 0$.

6.14 Integral Representation Formulas and Continuity Relations between Surface Electric Current Density and Surface Electric Charge Density.

As was pointed out in Section 3.2, the so-called Stratton-Chu formula (25) does not necessarily satisfy the Maxwell equations (4), so far as the additional conditions (26) are not taken into consideration. The conditions (26) which make the formula (25) represent fields are the conditions related to the continuity relation between surface electric current and charge densities. So, in this section, we shall study more about surface current and charge densities, and will introduce a new and simple condition (52) below which is equivalent to the continuity relation and which enables an integral representation formulas to really represent fields.

Suppose that S is an open boundary defined in Section 4.2, C is the periphery of S, $\tilde{n}$ is a normal vector on S and $\tilde{t}$ is a tangential vector of C. Set $\nabla^{(2)} = \nabla - \tilde{n}\partial/\partial n$ and $\tilde{P} = \tilde{t} \times \tilde{n}$. As was defined in Section 4.1, let $\{\}^+$ and $\{\}^-$ denote limiting values of $\{\}$ on S.

Let $\tilde{E}$ and $\tilde{H}$ be electromagnetic fields in a domain outside of S, and set $\tilde{J} = \tilde{n} \times \{(\tilde{H})^+ - (\tilde{H})^-\}$ and $\varpi = \tilde{n} \cdot \{(\tilde{E})^+ - (\tilde{E})^-\}$, then $\tilde{J}$ and ϖ, respectively, are called surface electric current density and surface electric charge density induced on S by the fields $\tilde{E}$ and $\tilde{H}$.

Then, as has been proved in [13] and [18], it follows that

$$\nabla^{(2)} \cdot \tilde{J} + i\omega\epsilon\varpi = 0 \tag{94}$$

Furthermore, from (94), it follows that

$$\int_S \{\nabla^{(2)} \cdot \tilde{J}(Q)\Psi(P',Q) + i\omega\epsilon\varpi(Q)\Psi(P',Q)\}dS_Q = 0 \tag{95}$$

where $\Psi(P',Q) = e^{-ikR}/4\pi R$, $R = \overline{P'Q}$ and $P' \notin S$.

(94), or (95), is called the continuity relation between surface electric current and charge densities.

Now, we shall prove the following theorems concerning the necessary and sufficient condition for an integral representation formula to represent electromagnetic fields, and also concerning continuity relations between surface electric and charge densities.

Theorem A

If $\tilde{J} = \tilde{n} \times \{(\tilde{H})^+ - (\tilde{H})^-\}$ is a surface electric current density on S induced by a field $\tilde{H}$, then, it should satisfy the following condition on the periphery C of S.

$$\tilde{J} \cdot \tilde{P} = 0 \tag{52}$$

Theorem B

The condition (52) and the conditions

$$\int_S \{\tilde{J} \cdot \nabla^{(2)}\Psi + \nabla^{(2)} \cdot \tilde{J}\Psi\}dS = 0 \tag{96}$$

$$\int_S \{(\tilde{J} \cdot \nabla)\nabla\Psi + \nabla^{(2)} \cdot \tilde{J}\nabla\Psi\}dS = 0 \tag{97}$$

are equivalent to each other.

Theorem C

Conversely, if $\tilde{J}$ is a tangential vector on S which satisfies (52) or (96) or (97) but otherwise arbitrary, and if $\tilde{E}$ and $\tilde{H}$ are defined in terms of $\tilde{J}$ by

$$\tilde{E}(P') = (i/\omega\epsilon)\int_S \{\nabla^{(2)}\cdot\tilde{J}(Q)\nabla_Q\Psi(P',Q) - k^2\Psi(P',Q)\tilde{J}(Q)\}dS_Q$$
$$\tilde{H}(P') = \int_S \tilde{J}(Q)\times\nabla_Q\Psi(P',Q)dS_Q \qquad (98)$$

then, $\tilde{E}$ and $\tilde{H}$ satisfy the Maxwell equations (4) at $P' \notin S$.

That is, each one of (52), (96) and (97) is the necessary and sufficient condition that (98) satisfies the Maxwell equations.

Furthermore, $\tilde{J}$ is the surface electric current density induced on S, by the fields defined by (98) in terms of $\tilde{J}$ itself. Moreover, $\tilde{J} = \tilde{n}\times\{(\tilde{H})^+ - (\tilde{H})^-\}$ and $\varpi = \tilde{n}\cdot\{(\tilde{E})^+ - (\tilde{E})^-\}$ calculated from (98), satisfy the continuity relation (94) mentioned above.

Also, it is shown that (98) is equivalent to

$$\tilde{E}(P') = \int_S \{(1/i\omega\epsilon)(\tilde{J}(Q)\cdot\nabla_Q)\nabla_Q\Psi(P',Q) - i\omega\mu\Psi(P',Q)\tilde{J}(Q)\}dS_Q \qquad (99)$$
$$\tilde{H}(P') = \int_S \tilde{J}(Q)\times\nabla_Q\Psi(P',Q)dS_Q$$

Theorem D

Each of the conditions (52), (96) and (97), is equivalent to the continuity relation (94).

Before going to the proof of these theorems, we shall introduce some lemmas.

Lemma 1

Let $\tilde{v}$ and Φ be arbitrary vector and scalar functions, respectively. Then, it is proved that[13],[18], $\int_S \nabla^{(2)}\cdot\tilde{v}dS = \oint_C \tilde{v}\cdot\tilde{P}ds$ and $\nabla^{(2)}\cdot(\Phi\tilde{v}) = \Phi\nabla^{(2)}\cdot\tilde{v} + \nabla^{(2)}\Phi\cdot\tilde{v}$ hold identically.

Combining these two equations, the following identical equation is obtained.

$$\int_S \{\Phi\nabla^{(2)}\cdot\tilde{v} + \tilde{v}\cdot\nabla^{(2)}\Phi\}dS = \oint_C \Phi\tilde{v}\cdot\tilde{P}ds \qquad (100)$$

Lemma 2

Let J and ϖ be surface electric current and charge densities on an open boundary S induced by electromagnetic fields $\tilde{E}$ and $\tilde{H}$ in a domain outside of S, then, it is proved that

$$\int_S \{\tilde{J}(Q) \cdot \nabla^{(2)} \Psi(P', Q) - i\omega\epsilon\varpi(Q)\Psi(P', Q)\} dS_Q = 0 \tag{101}$$

holds at any point $P' \notin S$.

Proof of (101)

Suppose that F is a union of closed surfaces introduced in Section 4.2, which encloses an open boundary S. Because F is a fictitious boundary, the Maxwell equations (4), which hold in a domain outside of S, hold on F as well. Consequently, we have $\tilde{u} = [\tilde{H} \times \nabla\Psi] - i\omega\epsilon\Psi\tilde{E} = [\tilde{H} \times \nabla\Psi] - \Psi\nabla \times \tilde{H} = -\nabla \times (\Psi\tilde{H})$ on F, which lead us, with the help of the Stokes theorem, to

$$(*) \qquad \int_F \{[\tilde{n} \times \tilde{H}] \cdot \nabla\Psi - i\omega\epsilon(\tilde{n} \cdot \tilde{E})\Psi\} dS = \int_F \tilde{n} \cdot \tilde{u} dS$$
$$= -\int_F \tilde{n} \cdot \nabla \times (\Psi\tilde{H}) dS = -\int_C \Psi\tilde{H} \cdot d\tilde{s}$$

which is zero because the periphery C of a closed surface F is empty. When $F = S^+ + S^- + S(\rho)$ shrinks to S, the sum of integrals on S^+ and S^- in $(*)$ is reduced, by virtue of the opposite directions of $\tilde{n}$ on S^+ and S^-, to (101), while that on $S(\rho)$ is shown to vanish by the same way as was taken in Section 6.13. [Q.E.D]

Note that (26) in Section 6.6 is similar to (101) but is different from it, because the range of integration of (26) is a closed surface while that of (101) is an open surface. Hence, the procedure which proved (101) is different from that proved (26).

Proof of Theorem A

If we set $\tilde{v}(Q) = \tilde{J}(Q)$ and $\Psi(Q) = \Psi(P', Q)$, $P' \notin S$, the identity (100) is reduced to

$$\int_S \{\Psi(P', Q)\nabla^{(2)} \cdot \tilde{J}(Q) + \tilde{J}(Q) \cdot \nabla\Psi(P', Q)\} dS_Q$$
$$= \oint_C \Psi(P', Q)\tilde{J} \cdot \tilde{P}(Q) ds_Q \tag{102}$$

On the other hand, the sum of (95) and (101) yields

$$\int_S \{\Psi(P',Q)\nabla^{(2)}\cdot\tilde{J}(Q)+\tilde{J}(Q)\cdot\nabla\Psi(P',Q)\}dS_Q=0 \tag{103}$$

From (102) and (103), it is shown that $\oint_C \Psi(P',Q)\tilde{J}\cdot\tilde{P}(Q)ds_Q = 0$ holds for all $P'\notin S+C$. Consequently, it is also shown that $\nabla_{P'}\cdot\oint_C \tilde{J}\cdot\tilde{P}(Q)\Psi(P',Q)ds_Q = -\oint_C \tilde{J}\cdot\tilde{P}(Q)\nabla_Q\Psi(P',Q)ds_Q = 0$ holds at any point $P'\notin S+C$, including P' which tends to $P\in C$. However, because of the singularity of $\nabla_Q\Psi(P',Q)$ at $P'=Q$, the last expression can not be finite so far as $\tilde{J}\cdot\tilde{P}(P)\neq 0$ at $P\in C$.

Thus, we have proved Theorem A. That is, (52) has been proved to be a necessary condition that $\tilde{E}$ and $\tilde{H}$ satisfy the Maxwell equations (4).

Proof of Theorem B

(96) is the same as (103) which we have proved above. It is easy to see that the gradient of (96) is reduced to (97), and the divergence of (97) is reduced to (96). Consequently, (96) and (97) are equivalent to each other.

On the other hand, if (52) holds, the identity (102) proves (96). Conversely, it has already been proved in the proof of Theorem A that (96) and (102) prove (52). As a consequence, (52), (96) and (97) are equivalent to each other.

Proof of Theorem C

If $\tilde{J}$ satisfies (52), then (97) holds, and $\tilde{E}$ and $\tilde{H}$ defined by (98) are reduced to those defined by (99).

On the other hand, since (99) is (28) with $\tilde{J}^*=0$, (99) satisfies the Maxwell equations (4). Hence (98) satisfies (4) as well. Thus the first assertion of Theorem C has been proved.

By the jump relation introduced by (79) in Section 6.10, second expression of (98) shows that (#) $\tilde{J}=\tilde{n}\times\{(\tilde{H})^+-(\tilde{H})^-\}$, which proves the second assertion of Theorem C.

On the other hand, we have another jump relation [13], which is

$$\begin{aligned}\tilde{n}(P)\cdot\{[\int_S \varpi(Q)\nabla_Q\Psi(P',Q)dS_Q]^+ - \\ [\int_S \varpi(Q)\nabla_Q\Psi(P',Q)dS_Q]^-\} = \varpi(P)\end{aligned} \tag{104}$$

With the help of (104) applied to (98), we have

$$\varpi(P) = \tilde{n}(P)\cdot\{[\tilde{E}(P')]^+ - [(\tilde{E}(P')]^-\} = (i/\omega\epsilon)\nabla_P^{(2)}\cdot\tilde{J}(P)$$

which, together with (#) above, proves the third assertion of Theorem C. [Q.E.D]

The proof of Theorem D has already been given above.

Note

(26) in a case of a closed boundary, and such as (96) in a case of an open boundary, enable the integral representation formula to really represent electromagnetic fields. However, they require surface integral. Contrary to them, (52), which enables the formulas (98) and (99) to really represent electromagnetic fields, requires only line integral on the periphery of an open boundary. This advantage is useful in a practical numerical analysis.

6.15 A "Simple Layer Potential" Satisfies Radiation Condition and Edge Condition.

Let L be an open boundary defined in Section 4, and set $u(P') = \int_L \Psi(P',Q)\tau(Q)ds_Q$, which has appeared in the text as a representation of a two-dimensional E wave. We may call this function a "simple layer potential" in a resemblance to a simple layer potential in potential theory.

It is obvious that $u(P')$ satisfies the Helmholtz equation (7) because $\Psi(P',Q)$ satisfies it when $P' \neq Q$. $u(P')$ is also proved to satisfy the radiation condition (13) in the same way as that proved (22) to satisfy (13) in Section 6.4. In this section, we shall show that $u(P')$ satisfies the edge condition (13).

If we set $\Psi(P',Q) = (1/4i)H_0^{(2)}(k\overline{P'Q}) = (-2\pi)^{-1}\log\overline{P'Q} + \Psi_0(P',Q)$, $\Psi_0(P',Q)$ is bounded even if $P' = Q$. So, the behavior of

$u(P')$ in a vicinity of an end point P_n^* of L is dominated by that of $V(P') = (1/2\pi)\int_L \{\partial\varphi(Q)/\partial s(Q)\} \log \overline{P'Q} ds_Q$, where $\partial\varphi(Q)/\partial s(Q) = -\tau(Q)$ is the tangential derivative of a function $\varphi(Q)$ with respect to the arc length s of L. In the following, we shall make use of the results in Muskhelishvili[5] on the theory of a Cauchy integral.

Consider a Cauchy integral on a complex z-plane; $\Phi(z) = (1/2\pi i)\int_L \varphi(\zeta)/(\zeta - z)d\zeta$, where ζ and z are complex variables corresponding to $Q \in L$ and $P' \notin L$, respectively. Then, assuming that $\varphi(\zeta)$ is real valued, and separating $\Phi(z)$ in its real and imaginary parts, we have $\mathrm{Im}\Phi(z) = V(P') - (1/2\pi)\sum(-1)^n\varphi(P_n^*)\log\overline{PP_n^*}$, where $\sum$ means the sum with respect to all end points P_n^* of L.

On the other hand, it is proved[5] that we have, in a vicinity of an end point P_m^*, $\Phi(z) = (-1)^m(1/2\pi i)\varphi(P_m^*)\log\overline{P'P_m^*} + \Psi_0(P')$ where $\Psi_0(P')$ is bounded at $P = P_m^*$. Combining these results, we have

$$\begin{aligned} V(P') =& \mathrm{Im}[(-1)^m(1/2\pi i)\varphi(P_m^*)\log\overline{P'P_m^*}] \\ &+ (1/2\pi)\sum(-1)^n\varphi(P_n^*)\log\overline{P'P_n^*} + \mathrm{Im}\Psi_0(P') \\ =& (1/2\pi)\sum_{n\neq m}(-1)^n\varphi(P_n^*)\log\overline{P'P_n^*} + \mathrm{Im}\Psi_0(P') \end{aligned}$$

which is bounded near and at any end point P_m^*. Hence we have, $u(P') = 0(1)$ and $\partial u(P')/\partial\rho = 0(\rho^{-\alpha})$ where $\rho = \overline{P'P_m^*}$ and $\alpha < 1$. It is easy to see that the same holds when $\varphi(Q)$ is complex.

6.16 Proof of the Theorem in Section 5.3 Which Asserts That U is Dense in $L_2(L)$.

A reader is referred, if necessary, to Section 6.18 for explanation of terminology and theorems.

Let L be an open boundary, $\{\varphi_n\}$ be a complete system in $L_2(L)$, and U be the set of all linear combinations of functions u_n, where u_n is the "elemental field function" defined by $u_n(P') = \int_L \Psi(P', Q)\varphi_n(Q)ds_Q$. We want to prove that U is dense in $L_2(L)$, that is, any element f of $L_2(L)$ is approximated by a pertinent linear combination of u_n as precisely as wanted. In other words, any element f of $L_2(L)$ is a limit of elements u_n of U. Again, in other words, the closure $\overline{U}$ coincides with $L_2(L)$.

Since our series expansion approach is based on the fact that we can approximate any given boundary data by a pertinent linear combination of u_n, this theorem is fundamentally important.

Assume that $L_2(L) \neq \overline{U}$, then there is an element f_0 such that $f_0 \in L_2(L)$ but $f_0 \notin \overline{U}$. Then, by virtue of the Hahn-Banach Extension theorem, it is shown that there exists a functional F such that $F(f_0) = 1$ and $F(u_n) = 0$ for all $u_n \in U$. Furthermore, in virtue of F. Riesz' theorem, there exists an element f of $L_2(L)$ such that $F(f_0) = (f, f_0)$ and $F(u_n) = (f, u_n)$.

By the definition, $(u_n, f) = \int_L \overline{f}(P) ds_p \int_L \Psi(P,Q)\varphi_n(Q) ds_Q$, which is because $\Psi(P,Q) = \Psi(Q,P)$, rewritten as $= \int_L \varphi_n(P)\overline{u}^*(P)$ ds_P, where $u^*(P) = \int_L \overline{\Psi}(P,Q) f(Q) ds_Q$. Hence, if $(u_n, f) = 0$ for all n, then we have $(\varphi_n, u^*) = 0$ for all n, which implies that $u^*(P) = 0$ because of the completeness of $\{\varphi_n\}$. This means that a function $\overline{u}^*(P') = \int_L \Psi(P',Q)\overline{f}(Q) ds_Q$, which satisfies the Helmholtz equation (7) and the radiation condition (13) in the outside of L, assumes the boundary value zero on L. Consequently, by virtue of the uniqueness theorem proved in Section 6.11, $u^*(P')$ is identically zero in the outside of L. Hence, $\frac{\partial}{\partial n(P)} \int_L \Psi(P',Q)\overline{f}(Q) ds_Q$ is also zero identically. Therefore, the limiting values of it from the positive as well as the negative sides of L, and accordingly the difference of them as well, are zero. Therefore, from the jump relations (73) in Section 6.10, we have $f(P) = 0$, which contradicts to the assumption that $(f, f_0) = 1$. Thus, we have proved that $L_2(L) = \overline{U}$. This implies that any element g of $L_2(L)$ is a limit of a sequence of elements of U, that is g is approximated by linear combinations of elements u_n of U as closely as we want.

6.17 Proof of the Theorem in Section 5.5 Which Asserts That $\tilde{U}$ is Dense in $\tilde{L}_2(S)$.

Let $\tilde{L}_2(S)$ be the set of all square integrable tangential vectors on S, $\{\tilde{J}_n\}$ be a complete system in $\tilde{L}_2(S)$, and $\tilde{E}_n$ and $\tilde{H}_n$ be "elemental field functions" defined by (61);

$$\tilde{E}_n(P') = (1/i\omega\epsilon) \int_S \{(\tilde{J}_n(Q) \cdot \nabla_Q)\nabla_Q \Psi(P',Q) + k^2 \Psi(P',Q)\tilde{J}_n(Q)\} dS_Q$$

$$\tilde{H}_n(P') = \int_S \tilde{J}_n(Q) \times \nabla_Q \Psi(P',Q) dS_Q$$

and set (62); $\tilde{u}_n(P) = \tilde{n}(P) \times \tilde{E}_n(P)$, $P \in S$. Suppose that $\tilde{U}$ is the set of all linear combinations of $\tilde{u}_n$, and that $\overline{\tilde{U}}$ is its closure.

Here, we shall prove that $\tilde{U}$ is dense in $\tilde{L}_2(S)$, that is, the closure $\overline{\tilde{U}}$ coincides with $\tilde{L}_2(S)$, and any function in $\tilde{L}_2(S)$ is approximated by a pertinent linear combination of $\tilde{u}_n$ as closely as wanted.

Assume that the contrary is true, then, there is an element $\tilde{f}_0$ such that $\tilde{f}_0 \in \tilde{L}(S)$ but $\tilde{f}_0 \notin \overline{\tilde{U}}$. In this case, by virtue of the Hahn-Banach Extension theorem, there exists a functional F such that $F(\tilde{f}_0) = 1$ and $F(\tilde{u}_n) = 0$ for all $\tilde{u}_n \in \tilde{U}$. On the other hand, by virtue of F. Riesz' theorem, there exists an element $\tilde{f} \in \tilde{L}_2(S)$ such that $F(\tilde{f}_0) = (\tilde{f}_0, \tilde{f})$ and $F(\tilde{u}_n) = (\tilde{u}_n, \tilde{f})$. By the definition,

$$(\tilde{u}_n, \tilde{f}) = \int_S \overline{\tilde{f}}(P) \cdot \tilde{u}_n(P) dS_P = (1/i\omega\epsilon) \int_S \overline{\tilde{f}}(P) \cdot [\tilde{n}(P) \times \int_S \{(\tilde{J}_n(Q) \cdot \nabla_Q) \nabla_Q \Psi(P,Q) + k^2 \Psi(P,Q) \tilde{J}_n(Q)\} dS_Q] dS_P$$

If we set $\tilde{j}(P) = \tilde{n}(P) \times \tilde{f}(P)$, and note that $\Psi(P,Q) = \Psi(Q,P)$ and $\nabla_P \Psi(P,Q) = -\nabla_Q \Psi(P,Q)$, then, by an elementary calculation, we have $\tilde{j}(P) \cdot (\tilde{J}_n(Q) \cdot \nabla_Q) \nabla_Q \Psi(P,Q) = \tilde{J}_n(Q) \cdot (\tilde{j}(P) \cdot \nabla_P) \nabla_P \Psi(P,Q)$. Hence, the last expression is reduced to $(\tilde{u}_n, \tilde{f}) = -\int_S \tilde{J}_n(Q) \cdot \tilde{e}(Q) dS_Q$, where $\tilde{e}(Q) = (1/i\omega\epsilon) \int_S \{(\overline{\tilde{j}}(P) \cdot \nabla_P) \nabla_P \Psi(P,Q) + k^2 \Psi(P,Q) \overline{\tilde{j}}(P)\} dS_P$.

Consequently, if $(\tilde{u}_n, \tilde{f}) = 0$ holds for all n, then, by the completeness of $\{\tilde{J}_n\}$ in $\tilde{L}_2(S)$, it follows that (#) $\tilde{n} \times \tilde{e}(P) = 0, \ P \in S$.

On the other hand, if we set $\tilde{h}(P') = \int_S \overline{\tilde{j}}(Q) \times \nabla_Q \Psi(P',Q) dS_Q$, then, it is obvious that they satisfy the Maxwell equations (4) in the outside of S. Furthermore, by the same way as that taken in Section 6.9, it is proved that they satisfy the radiation conditions (14). That is, $\tilde{e}(P')$ and $\tilde{h}(P')$ fulfill the conditions required by the uniqueness theorem in Section 6.11. Furthermore, by (#), $\tilde{e}$ assumes the boundary value zero on S. Consequently, by virtue of the uniqueness theorem, $\tilde{e}(P')$ and $\tilde{h}(P')$ vanish everywhere in the outside of S. Hence, the limiting values of $\tilde{n} \times \tilde{h}(P')$ in the limit as P' tends to $P \in S$ from the positive as well as the negative side of S, are zero. On the other hand, by the jump relations (79) in Section 6.10, $\tilde{j}$ is shown to be equal to the difference of these two limiting values, hence $\tilde{j} = \tilde{n} \times \tilde{f}$ must be zero. While $\tilde{f}$ is a tangential vector, hence $\tilde{n} \cdot \tilde{f} = 0$. Therefore, since $\tilde{f} = \tilde{n} \times [\tilde{f} \times \tilde{n}] + \tilde{n}(\tilde{n} \cdot \tilde{f})$, we have $\tilde{f} = \tilde{0}$, which result contradicts to the assumption $(\tilde{f}, \tilde{f}_0) = 1$. Thus we have proved that $\tilde{L}_2(S) = \overline{\tilde{U}}$, that is, $\overline{\tilde{U}}$ is dense in $\tilde{L}_2(S)$. This implies that any element $\tilde{G}$ of

$\tilde{L}_2(S)$ is approximated by a linear combination of $\tilde{u}_n$ as closely as we want.

6.18 Theorems and Explanation of Mathematical Terminology.

In this section, some terminology and theorems which appeared in the text will be explained.

Section 4

A curve which does not cross itself is simple. For instance, "8" is not simple, but "C" is simple. Similarly, a surface element which does not intersect with itself is simple.

A curve or a surface is smooth if its tangent at a point on it varies continuously with the movement of the point.

If a curve or a surface is divided into a finite number of parts, and if each of the parts is smooth, it is called piecewise smooth. "C" is smooth, but "V" is not smooth. However, "V" is piecewise smooth.

If a surface is two-sided, that is, it has a unique normal at every point on it, then it is called orientable. The so-called Mobius strip is the most famous one-sided surface and it is not orientable.

An unclosed surface element is simply connected if it has no hole in it, while it is n-ply connected if it has $n-1$ holes in it. The interior of a circle is simply connected, but a region between two coaxial circles is doubly connected.

Section 6

Open boundaries L and S were defined in Section 4. Definitions of $L_2(L)$, together with these of inner product, norm, orthogonality, limit, complete system and denseness, have been given in Section 5.2. The similar definitions on $\tilde{L}_2(S)$ have been given in Section 5.4.

The closure $\overline{U}$ of a set U in $L_2(L)$ is the sum of U and all elements of $L_2(L)$ such that any small neighborhood of it contains an element of U. In other words, $\overline{U}$ includes all functions in $L_2(L)$ such that every one of them is a limit of a sequence composed of elements of U.

A functional F is a correspondence which maps a function to a real number. In other words, a functional is a function of a function.

A function f is regular at a point P if f and its derivatives assume definite values at P. While, f is singular at P if it does not assume a definite value there.

A supremum of a set of numbers M, which is denoted as $\sup M$, is the minimum number among those numbers which are not less than any element of the set M.

A sequence of functions f_n is said to converge to a function f uniformly in a bounded closed domain D, if the supremum of the set of all values of $|f_n(P) - f(P)|$ with respect to $P \in D$ can be made as small as we want. If a domain D is unbounded, and if f_n converges to f uniformly in any closed subdomain of D, then a sequence f_n is said to converge to f in a wide sense in D.

Note that all of what have been stated above concerning $L_2(L)$ hold as they are in $\tilde{L}_2(S)$ as well, if scalar quantities in $L_2(L)$ are replaced by corresponding vector quantities in $\tilde{L}_2(S)$.

The following Theorems appeared without proof in Section 6.

Hahn – Banach Extension Theorem

Let M be a subspace of a normed space X such that $\overline{M} \neq X$. Then, there exists an element $f_0 \in X - \overline{M}$ and a Functional F such that $F(f_0) = 1$ and $F(u_n) = 0$ for all elements u_n of M.

F. Riesz′ Theorem

Let F be a functional on $L_2(L)$, then there exists an element $f \in L_2(L)$ such that $F(x) = (x, f)$ holds for any element x of $L_2(L)$. Similar holds for $\tilde{L}_2(S)$ as well.

The proof of these theorems are found in text books such as [19].

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