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A MULTISCALE MOMENT METHOD FOR SOLVING FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

C. Su

Dept. of Applied Math.
Northwestern Polytechnical University
Xian, Shaanxi 710072, P.R. China

T. K. Sarkar

Dept. of Electrical and Computer Engineering
Syracuse University Syracuse, NY 13244, USA

1. INTRODUCTION

As we know, Fredholm integral equation of the first kind is a kind of importance in many engineering fields. Several methods have been proposed, such as expansion method [1,2], regularization method [3], Backus-Gilbert method [4], Galerkin method [1], and the moment method [5], etc. But the moment method has played an important role in engineering computations. Fredholm integral equation of the first kind appears frequently in practice [6].

In general, the matrix constructed by the conventional moment method is dense, and so its solution is very time-consuming, particularly for large number of subsections. The major computational difficulty in implementing Galerkin's method is that virtually for all practical cases the inner products need to be evaluated numerically. In particular the task of evaluating the double integrals can be quite difficult and time-consuming for non-smooth kernel functions. To overcome the difficulties of large memory requirement and high computation time, many researchers have proposed the use of wavelet basis.

As we know, it is important to select a suitable basis function in numerical computation of integral equations and differential equations. Many kinds of basis functions have been proposed, such as triangular basis function, pulse basis function, polynomial basis function, spline and B-spline basis function. Recently, the wavelet basis function [7] or wavelet-like basis function [8] has been proposed to solve the numerical solutions of Fredholm equations and differential equation in one dimension. Steinbery et al. [9] used the wavelet expansions for the unknown current (function) in the moment method, which is expressed as a twofold summation of shifted and dilated forms of properly chosen basis function. Goswami et al. [10] used wavelets on a bounded interval to solve the first-kind integral equations in electromagnetic scattering problems.

Wang [11] proposed a hybrid method based on the wavelet expansion method and boundary element method. In his method, the unknown surface current is expanded in terms of a basis derived from periodic, orthogonal wavelet in interval [0,1].

Because of the local supports and vanishing moment properties of wavelets, many of the matrix elements are very small compared to the largest element, and hence can be neglected without significantly affecting the solution. Using moment method and subsequently a threshold procedure, the matrix constructed by these methods can be rendered sparse. Then, the linear equation with the sparse matrix is solved.

The objective of this paper is to propose an efficient method for solving Fredholm integral equation of the first kind from the point of view of reducing the order of the linear equation, rather than making the matrix sparse. First, a new method for approximating a function has been proposed based on multiscaling and wavelet-like basis. Second, by use of this kind of basis, the multiscale moment method for solving Fredholm integral equation of the first kind in one dimension has been proposed. Furthermore, the adaptive algorithm of the multiscale moment method has been presented according to the characteristics of the solution of the integral equation. Many of the numerical simulations are carried out to test the feasibility of the multiscale moment method and its adaptive algorithm.

2. FORMULA FOR BASIS FUNCTIONS BASED ON MULTISCALING TECHNIQUE

For reasons of simplicity, consider the problem of approximating a function $f(x)$ in the interval [0,1]. Suppose the 0'th approximation function $f_0(x)$ is defined as follows:

$$f_0(x) = f(x_{0,0})\phi_0(x) + f(x_{0,1})\phi_1(x) \tag{1.1}$$

where

$$\phi_0(x) = \phi_{0,1}(x) = \begin{cases} 1-x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases},$$
$$\phi_1(x) = \phi_{0,2}(x-1), \quad \phi_{0,2}(x) = \begin{cases} x-1 & x \in [-1,0] \\ 0 & \text{otherwise} \end{cases}$$
$$x_{0,0} = 0, \qquad x_{0,1} = 1$$

By multiscaling the interval on [0,1], the new interpolation node is $x_{1,1} = \frac{x_{0,0}+x_{0,1}}{2} = \frac{1}{2}$, and the 1'st approximation function $f_1(x)$ is defined as follows:

$$f_1(x) = f_0(x) + \tau_{1,1}\phi_{1,1}(x) \tag{1.2}$$

where $\phi_{1,1}(x) = \phi_{0,1}[2(x-x_{1,1})] + \phi_{0,2}[2(x-x_{1,1})]$. $\tau_{1,1}$ is an unknown coefficient.

Let us suppose $f_1(x_{1,1}) = f(x_{1,1})$, then

$$\tau_{1,1} = f(x_{1,1}) - f_0(x_{1,1}) = f(x_{1,1}) - \frac{1}{2}(f(x_{0,0}) + f(x_{0,1})) \tag{1.3}$$

By multiscaling the interval [0,1] again, the new interpolation nodes obtained are, $x_{2,1} = \frac{x_{0,0}+x_{1,1}}{2} = \frac{1}{4}$, $x_{2,1} = \frac{x_{1,1}+x_{0,1}}{2} = \frac{3}{4}$. The 2'nd approximation function $f_2(x)$ is generated as follows:

$$f_2(x) = f_1(x) + \tau_{2,1}\phi_{2,1}(x) + \tau_{2,2}\phi_{2,2}(x) \tag{1.4}$$

where $\phi_{2,1}(x) = \phi_{0,1}[2(x - x_{2,1})] + \phi_{0,2}[2(x - x_{2,1})]$, $\phi_{2,2}(x) = \phi_{0,1}[2(x - x_{2,2})] + \phi_{0,2}[2(x - x_{2,2})]$, $\tau_{2,1}$, $\tau_{2,2}$ are unknown coefficients.

Let us suppose $f_2(x_{2,1}) = f(x_{2,1})$, $f_2(x_{2,2}) = f(x_{2,2})$ then:

$$\tau_{2,1} = f(x_{2,1}) - f_2(x_{2,1}) = f(x_{2,1}) - \frac{1}{2}(f(x_{0,0}) + f(x_{1,1})),$$
$$\tau_{2,2} = f(x_{2,2}) - f_2(x_{2,2}) = f(x_{2,2}) - \frac{1}{2}(f(x_{0,1}) + f(x_{1,1}))$$

In general, by J-times multiscaling the interval [0,1] the new interpolation nodes generated are $\{x_{J,i} = \frac{1}{2^J} + \frac{i-1}{2^{J-1}},\ i = 1, 2, \ldots, 2^{J-1}\}$, the J'th approximation function $f_J(x)$ is defined as follows:

$$\begin{aligned} f_J(x) &= f_{J-1}(x) + \sum_{i=1}^{2^{J-1}} \tau_{j,i}\phi_{j,i}(x) \\ &= f_0(x) + \sum_{j=1}^{J} \sum_{i=1}^{2^{J-1}} \tau_{j,i}\phi_{j,i}(x) \end{aligned}$$

where $\phi_{j,i}(x) = \phi_{0,1}[2^j(x - x_{j,i})] + \phi_{0,2}[2^j(x - x_{j,i})]$, $\tau_{j,i}$ are unknown coefficients.

Let $f_J(x_{J,i}) = f(x_{J,i}),\ i = 1, 2, \ldots, 2^{J-1}$, then

$$\begin{aligned} \tau_{J,i} &= f(x_{J,i}) - f_{J-1}(x_{J,i}) = f(x_{J,i}) - f_0(x_{J,i}) - \sum_{j=1}^{J} \sum_{n=1}^{2^{J-1}} \tau_{j,n}\phi_{j,n}(x_{J,i}) \\ &= f(x_{J,i}) - \frac{1}{2}\left(f(x_{J,i} - \frac{1}{2^J}) + f(x_{J,i} - \frac{1}{2^J})\right) \end{aligned}$$

If $f(x)$ possesses a second order differentiability condition at $x_{J,i}$, then

$$\tau_{J,i} \approx -\frac{1}{2}\frac{1}{2^{J+1}} f''(x_{J,i}) \tag{1.5}$$

Apparently, this kind of functions $\{\phi_0(x), \phi_1(x), \phi_{J,i}(x);\ J = 1, 2, \ldots, i = 1, 2, \ldots, 2^{J-1}\}$ can be used as a set of basis functions in the interval [0,1]. And it is related to the triangular basis functions $\{\Psi_i(x)\}$.

For one multiscale, we have the following formula:

$$\begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 1 \\ 0 & 1 & -1/2 \end{pmatrix} \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_{1,1}(x) \end{pmatrix}$$

The transform matrix between original basis and new kind of basis is T:

$$T_1 = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 1 \\ 0 & 1 & -1/2 \end{pmatrix}$$

For two multiscale, the transform matrix can be written as:

$$T_2 = \begin{pmatrix} 1 & 0 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -/2 & 0 & -1/2 \end{pmatrix}$$

For three multiscale, the transform matrix can be written as:

$$T_3 = \begin{pmatrix} 1 & 0 & -1/2 & -1/2 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & 0 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1/2 & 0 & -1/2 & 0 & 0 & 0 & -1/2 \end{pmatrix}$$

The plots of $\phi_0(x)$, $\phi_1(x)$, $\phi_{1,1}(x)$, $\phi_{2,1}(x)$, $\phi_{2,2}(x)$, $\phi_{3,2}(x)$ are shown in Fig. 1.

In general, a function $f(x)$ in [0,1] can be approximated by choosing a scaled version of the triangular basis functions, as illustrated by $\{\phi_i(x);\ i = 0, 1, 2, \ldots, N\}$. The node points are at $\{x_{0,i} = \frac{i}{N} = ih;\ i = 0, 1, 2, \ldots, N,\ h = \frac{1}{N}\}$. The basis functions can be written as follows:

$$\begin{aligned} &\phi_0(x) = \phi_{0,1}(x) \\ &\phi_N(x) = \phi_{0,2}(x-1) \\ &\phi_i(x) = \phi_{0,1}(x - x_{0,i}) + \phi_{0,2}(x - x_{0,i}) \quad i = 1, 2, \ldots, N-1 \end{aligned}$$

where

$$\phi_{0,1}(x) = \begin{cases} 1 - Nx & \text{for } x \in [0, h] \\ 0 & \text{otherwise} \end{cases}$$
$$\phi_{0,2}(x) = \begin{cases} 1 + Nx & \text{for } x \in [-h, 0] \\ 0 & \text{otherwise} \end{cases}$$

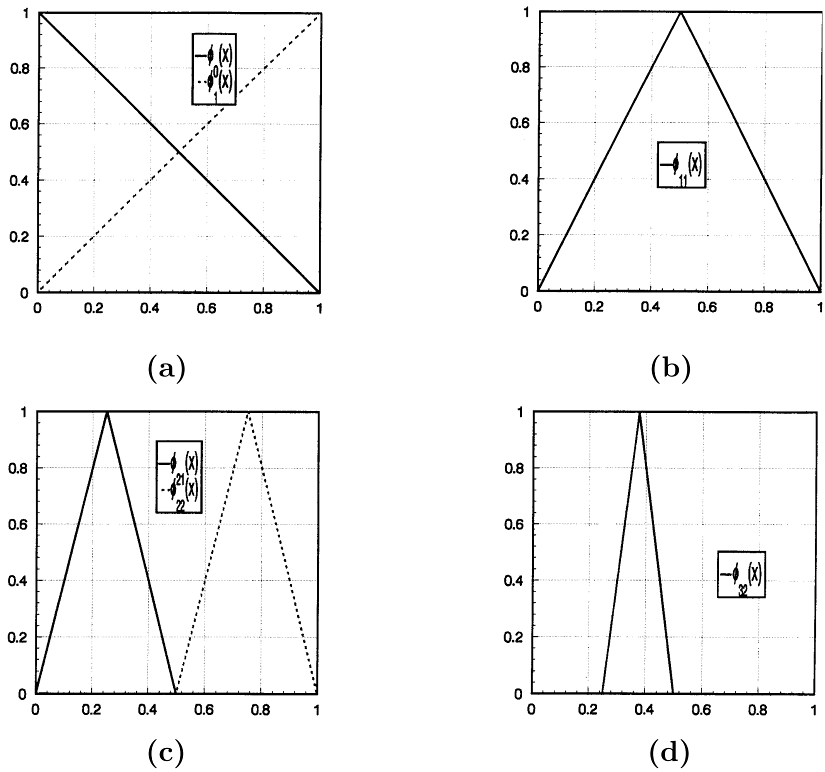


Figure 1.

The approximation for the function f_0 can be written as

$$f_0(x) = \sum_{n=0}^{N} f(x_{0,n})\phi_n(x) \tag{1.6}$$

By multiscaling the interval on [0,1], the new interpolation nodes are given by $x_{1,j} = \frac{x_{0,j-1}+x_{0,j}}{2} = \frac{1}{2N} + \frac{i-1}{N}$. The 1'st approximation function $f_1(x)$ is then defined as follows:

$$f_1(x) = f_0(x) + \sum_{i=1}^{N} \tau_{1,i}\phi_{1,i}(x) \tag{1.7}$$

where $\phi_{1,i}(x) = \phi_{0,1}[2(x - x_{1,i})] + \phi_{0,2}[2(x - x_{1,i})]$. $\tau_{1,j}$ are the unknown coefficients.

Let's suppose that $f_1(x_{1,i}) = f(x_{1,i})$, then:

$$\tau_{1,i} = f(x_{1,i}) - f_0(x_{1,i}) = f(x_{1,i}) - \frac{1}{2}(f(x_{0,i-1} + f(x_{0,i}))$$

By multiscaling the interval [0,1] again, the new interpolation nodes are given by $x_{2,i} = \frac{1}{4N} + \frac{i-1}{2N},\ (i = 1, 2, \ldots, 2N)$. The 2'nd approximation function $f_2(x)$ is written as follows

$$f_2(x) = f_0(x) + \sum_{i=1}^{N} \tau_{1,i}\phi_{1,i}(x) + \sum_{i=1}^{2N} \tau_{2,i}\phi_{2,i}(x) \tag{1.8}$$

where $\phi_{2,i}(x) = \phi_{0,1}[2^2(x - x_{2,i})] + \phi_{0,2}[2^2(x - x_{2,i})]$. $\tau_{2,j}$ are the unknown coefficients.

Let $f_2(x_{2,i}) = f(x_{2,i}),\ i = 1, 2, \ldots, 2^J N$, then

$$\tau_{2,i} = f_0(x_{2,i}) + \sum_{i=1}^{N} \tau_{1,i}\phi_{1,i}(x_{2,i}) \quad i = 1, 2, \ldots, 2^j N$$

By J-times multiscaling the interval [0,1] the new interpolation nodes are $\{x_{J,i} = \frac{1}{2^J N} + \frac{i-1}{2^{J-1}N},\ i = 1, 2, \ldots, 2^{J-1}N\}$. The J'th approximation function $f_J(x)$ is defined as follows:

$$\begin{aligned} f_J(x) &= f_{J-1}(x) + \sum_{i=1}^{2^{J-1}N} \tau_{J,i}\phi_{J,i}(x) \\ &= \sum_{n=0}^{N} f(x_{0,n})\phi_n(x) + \sum_{j=1}^{J} \sum_{i=1}^{2^{j-1}N} \tau_{j,i}\phi_{j,i}(x) \end{aligned} \tag{1.9}$$

where $\phi_{j,i}(x) = \phi_{0,1}[2^j(x - x_{j,i})] + \phi_{0,2}[2^j(x - x_{j,i})]$. $\tau_{J,i}$ are unknown coefficients.

Let $f_J(x_{J,i}) = f(x_{J,i}),\ i = 1, 2, \ldots, 2^{J-1}N$, then

$$\begin{aligned} \tau_{J,i} &= f(x_{J,i}) - f_{J-1}(x_{J,i}) \\ &= f(x_{J,i}) - f_0(x_{J,i}) - \sum_{j=1}^{J} \sum_{n=1}^{2^{j-1}N} \tau_{j,n}\phi_{j,n}(x_{J,i}) \\ &= f(x_{J,i}) - \frac{1}{2}\left(f(x_{J,i} - \frac{1}{2^J N}) + f(x_{J,i} + \frac{1}{2^J N}) \right) \end{aligned}$$

If $f(x)$ possesses the two-order continuous differentiable condition at $x_{J,i}$, then

$$\tau_{J,i} \approx -\frac{1}{2}\frac{1}{2^{J+1}N}f''(x_{J,i}) \tag{1.10}$$

We can prove that every function $f(x) \in \mathrm{C}[0,1]$ can be uniformly approximated by $\{f_J(x),\ J = 1,2,3,\ldots\}$, that is

$$f(x) = \sum_{i=0}^{N} f(x_{0,i})\phi_i(x) + \sum_{J=1}^{\infty}\sum_{i=1}^{2^{J-1}N} \tau_{J,i}\phi_{J,i}(x) \quad \text{on} \quad \mathrm{C}[0,1] \tag{1.11}$$

From the formula (1.5) and (1.10), we know that if $f(x)$ has the two-order continuous differentiability condition at $x_{J,i}$, the coefficients $\tau_{J,i}$ will be approximately zero as J increases. If $f(x)$ is linear at some interval in [0,1], $\tau_{J,i}$ will be zero. If $f(x)$ has a jump at x^* in [0,1], the coefficients $\tau_{J,i}$ near x^* will not decrease to zero. This property will be very useful in illustrating how to reduce the order of the linear equations constructed by the moment method, which will be discussed in the next section.

This new kind of basis has local support, but does not have vanishing moment properties unlike wavelets.

We also can deal with $f(x)$ being approximated by triangular functions based on non-uniform grid, and then use the multiscaling technique. The coefficients for the expansion and approximation of the discontinuous function utilizing a 5-time multiscale for the function

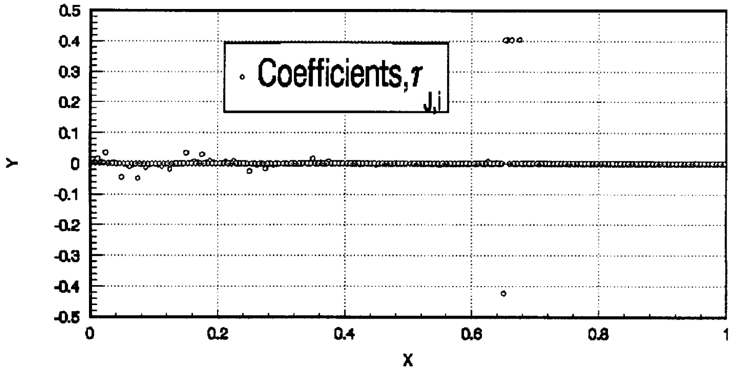
$$f(x) = \begin{cases} (0.5 - x)^2 \cos(10\pi x) & x \in [0, 0.6) \\ \sqrt{x} & x \in [0.6, 1] \end{cases}$$

is shown in Fig. 2a and Fig. 2b.

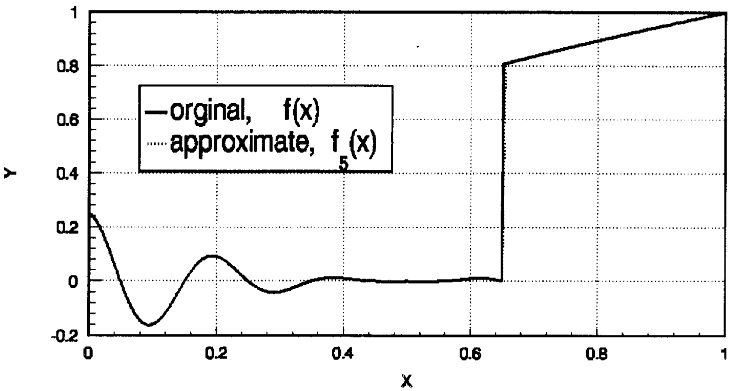
3. A MULTISCALE MOMENT METHOD FOR SOLVING INTEGRAL EQUATION

In the section, we discuss how to use a new kind of basis functions based on multiscaling the region Ω ([0,1]). We name this a multiscale moment method and use it to solve Fredholm integral equation of the first kind of the form

$$g(x) = \int_0^1 k(x,t)f(t)dt \quad x \in [0,1] \tag{2.1}$$



(a)



(b)

Figure 2.

From the above discussion, we know that every function $f(x) \in \mathrm{C}[0,1]$ can be written as follows:

$$f(x) = \sum_{n=0}^{N} f(x_{0,i})\phi_i(x) + \sum_{J=1}^{\infty} \sum_{i=1}^{2^{J-1}N} \tau_{J,i}\phi_{J,i}(x) \tag{2.2}$$

In the method of moment, we select $\{\phi_i(x),\, \phi_{J,k}(x)\}$ as the basis and testing functions. A finite set of functions is used, written as

$$f(x) \approx f_J(x) = \sum_{i=0}^{N} \tau_{0,i}\phi_i(x) + \sum_{j=1}^{J} \sum_{i=1}^{2^{j-1}N} \tau_{j,i}\phi_{j,i}(x) \tag{2.3}$$

By substituting the above formula into the integral equation, we obtain

$$g(x) = \sum_{i=0}^{N} \tau_{0,i} \int_0^1 k(x,t)\phi_i(t)dt + \sum_{j=1}^{J} \sum_{i=1}^{2^{j-1}N} \tau_{j,i} \int_0^1 k(x,t)\phi_{j,i}(t)dt \tag{2.4}$$

Also we select $\{\phi_i(x),\, \phi_{J,k}(x)\}$ as the set of weighting functions. And take the inner product of the above equation with the weighting functions and use the linearity of the inner product to obtain the following formula

$$\begin{aligned}
\int_0^1 g(x)\phi_m(x)dx &= \sum_{i=0}^{N} \tau_{0,i} \int_0^1 \int_0^1 k(x,t)\phi_i(t)\phi_m(x)dtdx \\
&\quad + \sum_{j=1}^{J} \sum_{i=1}^{2^{j-1}N} \tau_{j,i} \int_0^1 \int_0^1 k(x,t)\phi_{j,i}(t)\phi_m(x)dtdx \\
\int_0^1 g(x)\phi_{l,n}(x)dx &= \sum_{i=0}^{N} \tau_{0,i} \int_0^1 \int_0^1 k(x,t)\phi_i(t)\phi_{l,n}(x)dtdx \\
&\quad + \sum_{j=1}^{J} \sum_{i=1}^{2^{j-1}N} \tau_{j,i} \int_0^1 \int_0^1 k(x,t)\phi_{j,i}(t)\phi_{l,n}(x)dtdx
\end{aligned}$$

$$m = 0, 1, 2, \ldots, N, \quad l = 1, 2, \ldots, J, \quad n = 1, 2, \ldots, 2^{l-1}N$$

The set of equations can be written in a matrix form as

$$\begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_J \end{pmatrix} = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,J} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,J} \\ \vdots & \vdots & \ddots & \vdots \\ A_{J,0} & A_{J,1} & \cdots & A_{J,J} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_J \end{pmatrix}$$

where $X_0 = (\tau_{0,0},\, \tau_{0,1}, \ldots,\, \tau_{0,N})^T$, $X_j = (\tau_{j,1},\, \tau_{j,2}, \ldots,\, \tau_{j,2^{j-1}N})^T$, $j = 1, 2, \ldots, J)$

$$F_0(i) = \int_0^1 g(x)\phi_i(x)dx \quad i = 0, 1, 2, \ldots, N$$

$$F_j(i) = \int_0^1 g(x)\phi_{j,i}(x)dx \quad j = 1, 2, \ldots, J, \quad i = 1, 2, \ldots, 2^{j-1}N$$

$$A_{0,0}(i,k) = \int_0^1 \int_0^1 k(x,t)\phi_i(x)\phi_k(x)dxdt$$
$$i = 0, 1, \ldots, N, \quad k = 0, 1, \ldots, N$$

$$A_{0,j}(i,k) = \int_0^1 \int_0^1 k(x,t)\phi_i(x)\phi_{j,k}(x)dxdt$$
$$i = 0, 1, \ldots, N, \quad k = 1, 2, \ldots, 2^{j-1}N$$

$$A_{j,0}(k,i) = \int_0^1 \int_0^1 k(x,t)\phi_i(x)\phi_{j,k}(x)dxdt$$
$$i = 0, 1, \ldots, N, \quad k = 1, 2, \ldots, 2^{j-1}N$$

$$A_{j,l}(i,k) = \int_0^1 \int_0^1 k(x,t)\phi_{j,i}(x)\phi_{l,k}(x)dxdt$$
$$i = 0, 1, \ldots, 2^{l-1}N, \quad k = 1, 2, \ldots, 2^{j-1}N$$

The expressions for $F_0(i)$, $F_J(i)$, $A_{0,0}(i,k)$, $A_{0,J}(i,k)$, $A_{J,0}(k,i)$, $A_{J,l}(i,k)$ can be written as follows:

$$F_0(i) = \begin{cases} \displaystyle\int_0^{\frac{1}{N}} f(t + \frac{i}{N})(1 - Nt)dt & i = 1 \\ \displaystyle\int_0^{\frac{1}{N}} f(t + \frac{i}{N})(1 - Nt)dt + \\ \qquad \displaystyle\int_{-\frac{1}{N}}^{0} f(t + \frac{i}{N})(1 + Nt)dt & i = 1, 2, \ldots, N-1 \\ \displaystyle\int_{-\frac{1}{N}}^{0} f(t + \frac{i}{N})(1 + Nt)dt & i = N \end{cases}$$

$$\begin{aligned} F_J(i) = {} & \frac{1}{2^J} \int_0^{\frac{1}{N}} f\left(\frac{1}{2^J}(t + \frac{2i-1}{N})\right)(1 - Nt)dt \\ & + \frac{1}{2^J} \int_{-\frac{1}{N}}^{0} f\left(\frac{1}{2^J}(t + \frac{2i-1}{N})\right)(1 + Nt)dt \quad i = 1, 2, \ldots, 2^{J-1}N \end{aligned}$$

$$A_{0,0}(i,m) = \int_0^{\frac{1}{N}} (1-Nx)dx \int_0^{\frac{1}{N}} k(x+\frac{m}{N}, t+\frac{i}{N})(1-Nt)dt$$
$$+ \int_{-\frac{1}{N}}^{0} (1+Nx)dx \int_0^{\frac{1}{N}} k(x+\frac{m}{N}, t+\frac{i}{N})(1-Nt)dt$$
$$+ \int_0^{\frac{1}{N}} (1-Nx)dx \int_{-\frac{1}{N}}^{0} k(x+\frac{m}{N}, t+\frac{i}{N})(1+Nt)dt$$
$$+ \int_{-\frac{1}{N}}^{0} (1+Nx)dx \int_{-\frac{1}{N}}^{0} k(x+\frac{m}{N}, t+\frac{i}{N})(1+Nt)dt$$
$$(i = 1, 2, \dots, N-1, \;\; m = 1, 2, \dots, N-1)$$

$$A_{0,J}(i,m)$$
$$= \frac{1}{2^J} \int_0^{\frac{1}{N}} (1-Nx)dx \int_0^{\frac{1}{N}} k(x+\frac{m}{N}, \frac{1}{2^J}(t+\frac{2i-1}{N}))(1-Nt)dt$$
$$+ \frac{1}{2^J} \int_{-\frac{1}{N}}^{0} (1+Nx)dx \int_0^{\frac{1}{N}} k(x+\frac{m}{N}, \frac{1}{2^J}(t+\frac{2i-1}{N}))(1-Nt)dt$$
$$+ \frac{1}{2^J} \int_0^{\frac{1}{N}} (1-Nx)dx \int_{-\frac{1}{N}}^{0} k(x+\frac{m}{N}, \frac{1}{2^J}(t+\frac{2i-1}{N}))(1+Nt)dt$$
$$+ \frac{1}{2^J} \int_{-\frac{1}{N}}^{0} (1+Nx)dx \int_{-\frac{1}{N}}^{0} k(x+\frac{m}{N}, \frac{1}{2^J}(t+\frac{2i-1}{N}))(1+Nt)dt$$
$$(i = 1, 2, \dots, 2^{J-1}N, \;\; m = 1, 2, \dots, N-1)$$

$$A_{J,0}(i,m)$$
$$= \frac{1}{2^J} \int_0^{\frac{1}{N}} (1-Nx)dx \int_0^{\frac{1}{N}} k(\frac{1}{2^J}(x+\frac{2m-1}{N}), t+\frac{i}{N})(1-Nt)dt$$
$$+ \frac{1}{2^J} \int_{-\frac{1}{N}}^{0} (1+Nx)dx \int_0^{\frac{1}{N}} k(\frac{1}{2^J}(x+\frac{2m-1}{N}), t+\frac{i}{N})(1-Nt)dt$$
$$+ \frac{1}{2^J} \int_0^{\frac{1}{N}} (1-Nx)dx \int_{-\frac{1}{N}}^{0} k(\frac{1}{2^J}(x+\frac{2m-1}{N}), t+\frac{i}{N})(1+Nt)dt$$
$$+ \frac{1}{2^J} \int_{-\frac{1}{N}}^{0} (1+Nx)dx \int_{-\frac{1}{N}}^{0} k(\frac{1}{2^J}(x+\frac{2m-1}{N}), t+\frac{i}{N})(1+Nt)dt$$
$$(m = 1, 2, \dots, 2^{J-1}N, i = 1, 2, \dots, N-1)$$

$$
\begin{aligned}
&A_{J,l}(i,m)\\
&= \frac{1}{2^{J+l}} \int_0^{\frac{1}{N}} (1-Nx)dx \int_0^{\frac{1}{N}} k(\frac{1}{2^J}(x+\frac{2m-1}{N}), \frac{1}{2^l}(t+\frac{2i-1}{N}))\cdot\\
&\qquad (1-Nt)dt\\
&+ \frac{1}{2^{J+l}} \int_{-\frac{1}{N}}^{0} (1+Nx)dx \int_0^{\frac{1}{N}} k(\frac{1}{2^J}(x+\frac{2m-1}{N}), \frac{1}{2^l}(t+\frac{2i-1}{N}))\cdot\\
&\qquad (1-Nt)dt\\
&+ \frac{1}{2^{J+l}} \int_0^{\frac{1}{N}} (1-Nx)dx \int_{-\frac{1}{N}}^{0} k(\frac{1}{2^J}(x+\frac{2m-1}{N}), \frac{1}{2^l}(t+\frac{2i-1}{N}))\cdot\\
&\qquad (1+Nt)dt\\
&+ \frac{1}{2^{J+l}} \int_{-\frac{1}{N}}^{0} (1+Nx)dx \int_{-\frac{1}{N}}^{0} k(\frac{1}{2^J}(x+\frac{2m-1}{N}), \frac{1}{2^J}(t+\frac{2i-1}{N}))\cdot\\
&\qquad (1+Nt)dt\\
&\quad (m=1,2,\ldots,2^{J-1}N, i=1,2,\ldots,2^{l-1}N)
\end{aligned}
$$

The scheme of solving the above matrix equation is as follows:

Step 1: By utilizing the conjugate gradient method, we solve the equation $A_{0,0}X_0 = F_0$. The result is expressed as X_0^*.

Step 2: solve the equation $\begin{pmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$, by the conjugate gradient method. The initial guess $\begin{pmatrix} X_0 \\ X_1 \end{pmatrix}^{(0)}$ can be selected as $\begin{pmatrix} X_0^* \\ \vec{0} \end{pmatrix}$. The computed result is expressed as $\begin{pmatrix} X_0^* \\ X_1^* \end{pmatrix}$.

Step 3: solve the equation $\begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} \\ A_{1,0} & A_{1,1} & A_{1,2} \\ A_{2,0} & A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \end{pmatrix}$, by the conjugate gradient method utilizing the initial guess $\begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}^{(0)}$ for $\begin{pmatrix} X_0^* \\ X_1^* \\ \vec{0} \end{pmatrix}$. The result is expressed as $\begin{pmatrix} X_0^* \\ X_1^* \\ X_2^* \end{pmatrix}$. This procedure continues until the original equation is solved by satisfying some prescribed degree of accuracy.

Some Notes:

(1) This new method is a kind of moment method, but is different from the conventional MM. The coefficient matrix of the two methods has the following relation:

$$\begin{pmatrix} (\phi_1,\phi_1) & \dots & (\phi_1,\phi_N) \\ \vdots & \ddots & \vdots \\ (\phi_N,\phi_1) & \dots & (\phi_N,\phi_N) \end{pmatrix} = T \begin{pmatrix} (\psi_1,\psi_1) & \dots & (\psi_1,\psi_N) \\ \vdots & \ddots & \vdots \\ (\psi_N,\psi_1) & \dots & (\psi_N,\psi_N) \end{pmatrix} T'$$

where T is the transform matrix between original basis $\{\phi_i(x)\}$ and new kind of basis $\{\psi_i(x)\}$, that is

$$\begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{pmatrix} = T \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}$$

$$(f(x), g(t)) = \int_0^1 dx \int_0^1 k(x,t) f(x) g(t) dt$$

(2) This new method is different from the method used to solve the integral equations by use of the multigrid method. But it is like the wavelet method of solving the Fredholm equation.
(3) When it is desired to increase the scale, one needs to compute only the terms introduced by the new scale. But if the conventional MM or multigrid method is adapted, all of the elements of the coefficients matrix need to be computed again.
(4) The initial guess can be chosen easily.

4. ADAPTIVE ALGORITHM OF MULTISCALE MOMENT METHOD

From the above discussion, it is known that if the solution of the integral equation is smooth in some region, many of the coefficients $\{\tau_{J,i}\}$ will be near zero. So we can reduce the size of the linear equation in the moment method. For every scale, the order of the linear equation formed by the multiscale moment method will be changed according to the coefficients $\{\tau_{J,i}\}$, as some of them approach zero. This method is called an adaptive multiscale moment method. The scheme of the adaptive multiscale moment method used to solve the Fredholm integral equation of the first kind is given as follows:

Step 1: By use of the conjugate gradient method, we solve the equation $A_{0,0}X_0 = F_0$. The result is expressed as X_0^*.
Step 2: Using an interpolation technique, estimate the coefficients $\{\tau_{1,i}\}$ of the function $f_0(x)$ based on $(x_{0,i}, X_0^*(x))$ or $(x_{0,i}, f_0(x_{0,i}))$. If $|\tau_{1,i}| \le \varepsilon$ (threshold parameter) for $\{i = 1, \ldots, N\}$, the corresponding array and column of the matrix $\begin{pmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{pmatrix}$ will be omitted. After reducing the elements of the original matrix and getting an initial guess according to X_0^* and $\{\tau_{1,i}\}$, the solution of the linear equations can be obtained by use of the conjugate gradient method. The result of the original linear equation is expressed as $\begin{pmatrix} X_0^* \\ X_1^* \end{pmatrix}$, where X_1^* maybe contain some zero elements.
Step 3: Using an interpolation technique, estimate the coefficients $\{\tau_{2,i}\}$ of the function $f_1(x)$ based on $(x_{0,i}, f_1(x_{0,i}))$ and $(x_{1,i}, f_1(x_{1,i}))$. If $|\tau_{2,i}| \le \varepsilon$ for $\{i = 1, ..., 2N\}$ and $|\tau_{1,i}| \le \varepsilon$ for $\{i = 1, \ldots, N\}$, the corresponding array and column of the matrix $\begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} \\ A_{1,0} & A_{1,1} & A_{1,2} \\ A_{2,0} & A_{2,1} & A_{2,2} \end{pmatrix}$ will be omitted. After reducing the orders of the original matrix and getting an initial guess according to $\begin{pmatrix} X_0^* \\ X_1^* \end{pmatrix}$ and $\{\tau_{2,i}\}$, the solution of the linear equation can be obtained by use of the conjugate gradient method. The result of the original linear equation is expressed as $\begin{pmatrix} X_0^* \\ X_1^* \\ X_2^* \end{pmatrix}$, where X_2^* may contain some zero elements.

This procedure continues until some error criteria is met.

In the numerical computation the use of interpolation to estimate the coefficients, we adapt the following method:

For that points in the middle of the region, where the known data is $\{f(-\frac{3}{2}h), f(-\frac{1}{2}h), f(\frac{1}{2}h), f(\frac{3}{2}h)\}$, the value for $f(0)$ can be computed from the following formula by using a cubic polynomial approximation as

$$f(0) = \frac{[f(-\frac{1}{2}h) + f(\frac{1}{2}h)] - [f(-\frac{3}{2}h) + f(\frac{3}{2}h)]}{6}$$

For that point to the left of the region, suppose the known data is $\{f(-\frac{1}{2}h), f(\frac{1}{2}h), f(\frac{3}{2}h)\}$, the value $f(0)$ can be computed from the following formula according to the quadratic polynomial approximation

$$f(0) = \frac{[f(-\frac{1}{2}h) + f(\frac{1}{2}h)]}{2} - \frac{[2f(-\frac{1}{2}h) + f(\frac{3}{2}h)]}{3}$$

For the point to the right of the region, suppose the known data is $\{f(-\frac{3}{2}h), f(-\frac{1}{2}h), f(-\frac{1}{2}h)\}$, the value $f(0)$ can be computed from the following formula according to quadratic polynomial approximation

$$f(0) = \frac{[f(-\frac{1}{2}h) + f(\frac{1}{2}h)]}{2} - \frac{[2f(\frac{1}{2}h) + f(-\frac{3}{2}h)]}{3}$$

So interpolation can be used to estimate the coefficients, $\{\tau_{j,i}\}$.

5. NUMERICAL SIMULATI0NS

In order to test the feasibility of the multiscaling moment method for solving Fredholm integral equation of the first kind in one dimension, two kinds of kernel functions of the integral equation will be considered. The kernel functions, the exact solution and the source functions are listed in the following table.

$f(x)$	$g(x)$
1	$2(\sqrt{x} + \sqrt{1-x})$
x^2	$\frac{16}{15}x^2\sqrt{x} + \frac{2}{15}\sqrt{1-x}(3+4x+8x^2)$
$\begin{cases} ½ & x \in [0,0.5) \\ 1 & x \in [0.5,1] \end{cases}$	$\begin{cases} \sqrt{x} + 2\sqrt{1-x} - \sqrt{0.5-x} & x \in [0,0.5) \\ \sqrt{x} + 2\sqrt{1-x} + \sqrt{x-0.5} & x \in [0.5,1] \end{cases}$
$\begin{cases} 0.5 & x \in [0,0.5) \\ 2x & x \in [0.5,1] \end{cases}$	$\begin{cases} (\sqrt{x} + \sqrt{0.5-x}) + \frac{4}{3}\{\sqrt{1-x}(2x+1) - \sqrt{0.5-x}(2x+0.5)\} & x \in [0,0.5) \\ (\sqrt{x} + \sqrt{x-0.5}) + \frac{4}{3}\{\sqrt{1-x}(2x+1) + \sqrt{x-0.5}(2x+0.5)\} & x \in [0.5,1] \end{cases}$

Table 1a. $k(x,t) = \frac{1}{\sqrt{|x-t|}}$.

$f(x)$	$g(x)$
$\frac{4}{\ln 4}\frac{1}{\sqrt{1-(2x-1)^2}}$	2π
$\frac{4(2x-1)}{\sqrt{1-(2x-1)^2}}$	$2\pi(2x-1)$

Table 1b. $k(x,t) = -\ln|x-t|$.

The orders and index of the condition number of the system of linear equations of the multiscale moment method is given in Table 2a and Table 2b for different scales and initial divisions, respectively for the kernel functions $k(x,t) = \frac{1}{\sqrt{|x-t|}}$ and $k(x,t) = -\ln|x-t|$.

	0-scale	1-scale		2-scale		3-scale		4-scale		5-scale	
OLE	ICN	OLE	ICN	OLE	ICN	OLE	ICN	OLE	ICN	OLE	ICN
9	.40802E-1	17	.51886E-2	33	.15146E-2	65	.54638E-3	129	.16465E-4	257	.52637E-4
17	.31890E-1	33	.37184E-2	65	.10987E-2	129	.37662E-3	257	.12023E-3	513	.37955E-4
33	.27612E-1	65	.26804E-2	129	.78928E-3	257	.26275E-3	513	.86503E-4	1025	.27116E-4
65	.25981E-1	129	.19315E-2	257	.56305E-3	513	.18451E-3	1025	.61739E-4	2049	.19277E-4

Table 2a.

	0-scale	1-scale		2-scale		3-scale		4-scale		5-scale	
OLE	ICN	OLE	ICN	OLE	ICN	OLE	ICN	OLE	ICN	OLE	ICN
9	.40802E-1	17	.51886E-2	33	.15146E-2	65	.54638E-3	129	.16465E-4	257	.52637E-4
17	.31890E-1	33	.37184E-2	65	.10987E-2	129	.37662E-3	257	.12023E-3	513	.37955E-4
33	.27612E-1	65	.26804E-2	129	.78928E-3	257	.26275E-3	513	.86503E-4	1025	.27116E-4
65	.25981E-1	129	.19315E-2	257	.56305E-3	513	.18451E-3	1025	.61739E-4	2049	.19277E-4

Table 2b.

(where OLE is defined as the order of linear equations, ICN is equal to one over the condition number.) For the case (a) , $N = 8$ and the exact solution function $f^*(x) = 1$, the solutions of linear equations and the solution function $f(x)$ for different scales is shown in Fig. 3a and Fig. 3b.

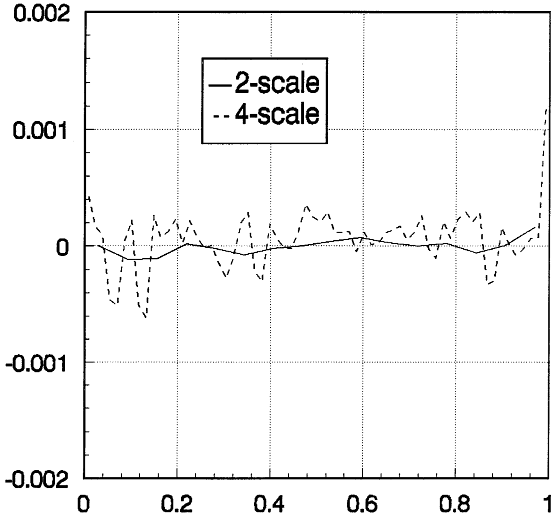


Figure 3a. The solution of the coefficients in different scales.

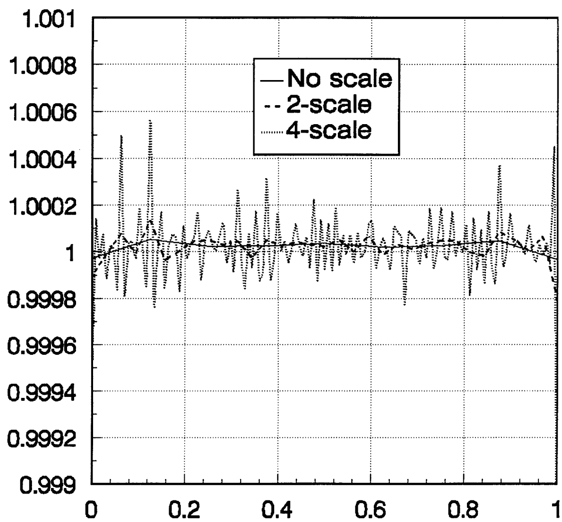


Figure 3b. The solution of the equation in different scales. Initial number of nodes, $N = 8$.

For the case (a) with $N = 16$, the exact solution $f^*(x) = x^2$, the solutions of linear equations and the solution $f(x)$ on different scales is shown in Fig. 4a and Fig. 4b.

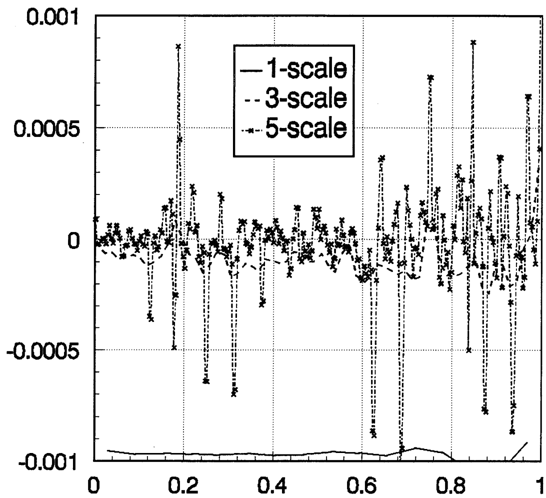


Figure 4a. The solution of the coefficients in different scales.

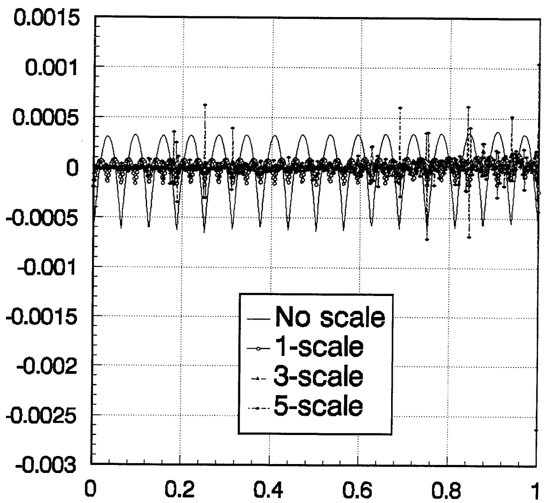


Figure 4b. The difference between the exact solution and the solutions of equation in different scales. Initial number of nodes, $N = 16$.

For the case (a) with $N = 32$, the exact solution $f^*(x) = \begin{cases} 1 & x \in [1, 0.5) \\ \frac{1}{2} & x \in [0, 0.5] \end{cases}$, the solutions of linear equations, and the solution $f(x)$ on different scales is shown in Fig. 5a and Fig. 5b.

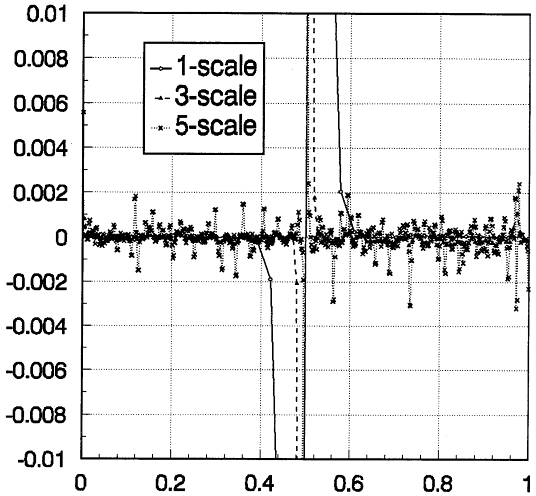


Figure 5a. The solution of the coefficients in different scales.

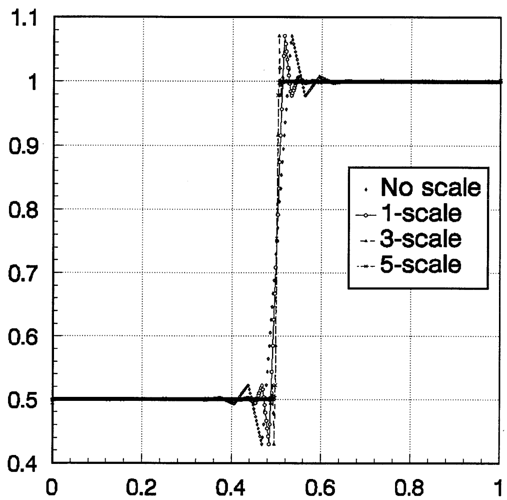


Figure 5b. The solution of the equation on different scales, initial number of nodes, $N = 32$.

For the case (a) with $N = 64$, the exact solution $f^*(x) = \begin{cases} 2x & x \in [0, 0.5) \\ 0.1 & x \in [0, 0.5] \end{cases}$, the solutions of linear equations and the solution $f(x)$ on different scales is shown in Fig. 6a and Fig. 6b.

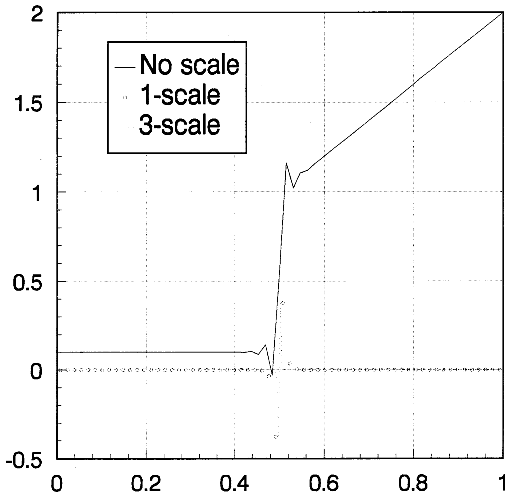


Figure 6a. The solution of the coefficients in different scales.

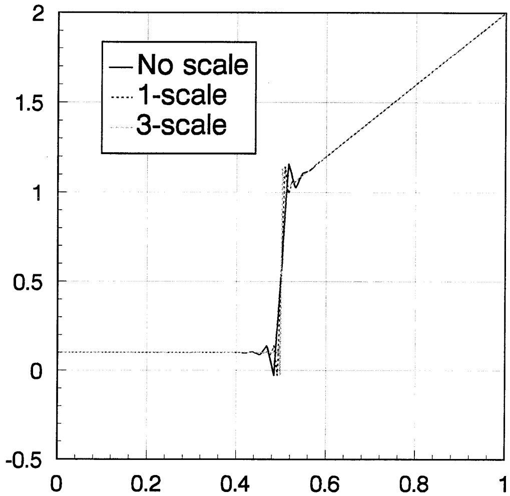


Figure 6b. The solution of the equation on different scales, initial number of nodes, $N = 64$.

For the case (b) with $N = 64$, the source function $g(x) = 2\pi$, the solutions of linear equations and the solution $f(x)$ on different scales is shown in Fig. 7a and Fig. 7b.

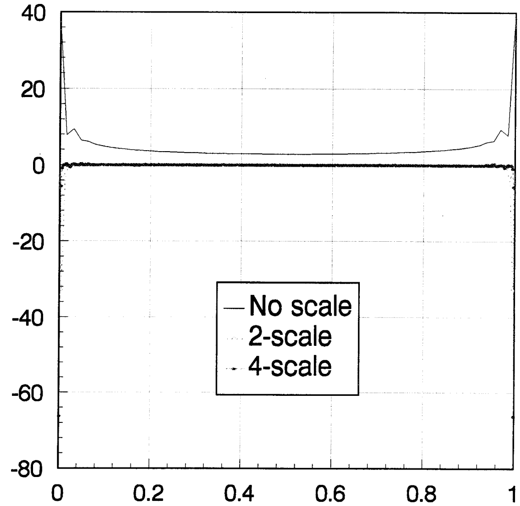


Figure 7a. The solution of the coefficients in different scales.

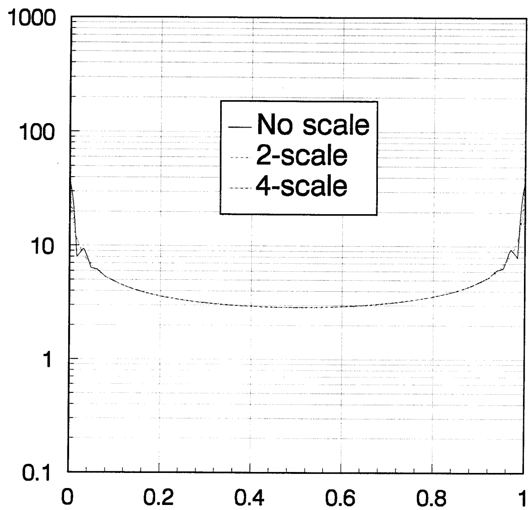


Figure 7b. The solution of the equation on different scales, initial number of nodes, $N = 64$.

For the case (b) with $N = 16$, the source function $g(x) = 2\pi(2x - 1)$, the solutions of linear equations and the solution $f(x)$ on different scales is shown in Fig. 8a and Fig. 8b.

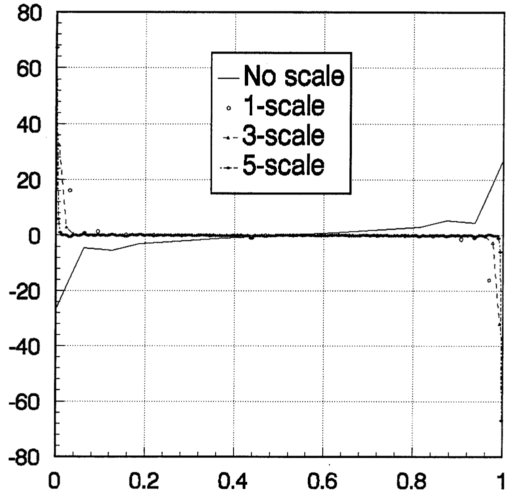


Figure 8a. The solution of the coefficients in different scales.

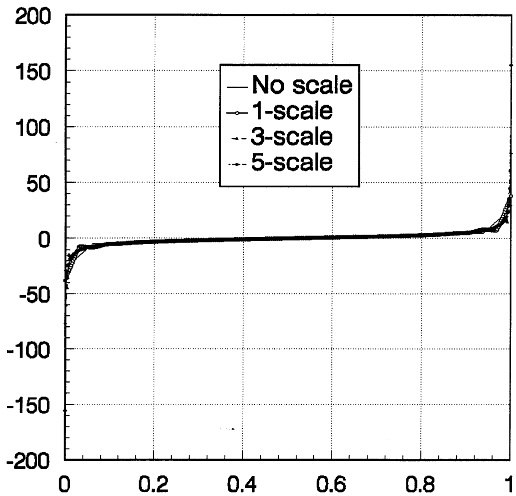


Figure 8b. The solution of the equation on different scales, initial number of nodes, $N = 16$.

Now the adaptive multiscale moment method is adapted to solve the above integral equations. For the case (a), the exact solution $f^*(x) = \begin{cases} 1 & x \in [1, 0.5) \\ \frac{1}{2} & x \in [0, 0.5] \end{cases}$, for different initial divisions and threshold values $\varepsilon = 0.0001$, the orders and the index of the condition number for the system of linear equations for the adaptive multiscale moment method is given in Table 3 for different scales. The solutions of linear equations and the solution function $f(x)$ on different scales for the case $N = 32$ is shown in Fig. 9a and Fig. 9b.

0-scale		1-scale		2-scale		3-scale		4-scale		5-scale	
OLE	ICN	OA	ICN	OA	ICN	OA	ICN	OA	ICN	OA	ICN
9	.40802E-1	17	.51886E-2	32	.15707E-2	43	.72116E-3	58	.27438E-2	76	.12979E-3
17	.31890E-1	33	.37184E-2	44	.14383E-2	55	.54715E-3	60	.23478E-3	81	.11821E-3
33	.27612E-1	48	.37050E-2	56	.11536E-2	69	.41674E-3	78	.21633E-3	99	.74436E-4
65	.25981E-1	80	.30138E-2	86	.10454E-2	101	.33927E-3	116	.33927E-3	142	.54164E-4

Table 3.

(where OA is defined as the order of the linear equation of adaptive multiscale moment method, ICN is equal to one over the condition number correspond to the matrix)

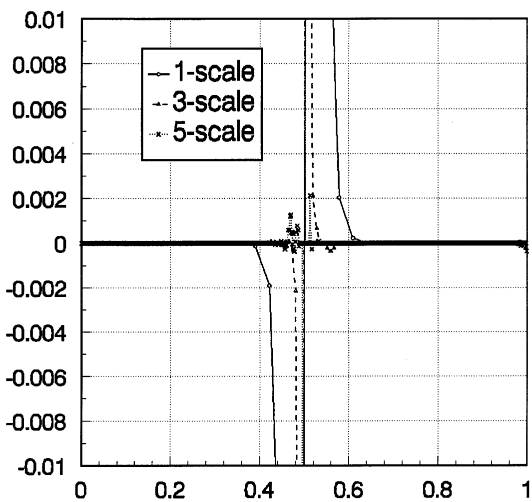


Figure 9a. The solution of the coefficients in different scales.

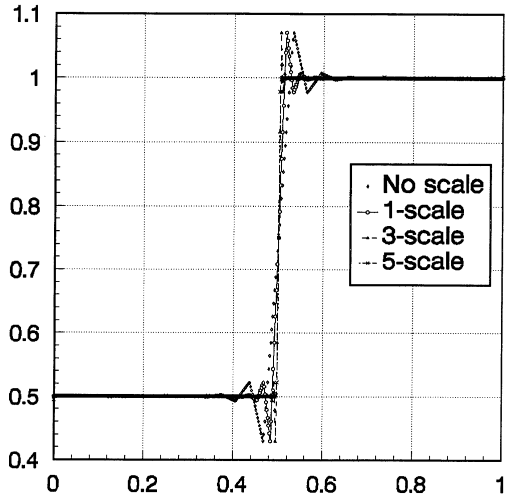


Figure 9b. The solution of the equation on different scales. Initial number of nodes, $N = 32$.

For the case (b), the source function $g(x) = 2\pi$, different initial divisions threshold $\varepsilon = 0.0001$, the orders and index of the condition number of linear equations of multiscale moment method is given in Table 4 for different scales. The solutions of linear equations and the solution function $f(x)$ on different scales in the case of $N = 32$ is shown in Fig. 10a and Fig. 10b.

0-scale		1-scale		2-scale		3-scale		4-scale		5-scale	
OLE	ICN	OA	ICN	OA	ICN	OA	ICN	OA	ICN	OA	ICN
9	.17742E-1	17	.16925E-2	33	.32166E-3	55	.93916E-4	78	.23773E-4	115	.52359E-5
17	.90134E-2	33	.81475E-3	60	.17949E-3	101	.47401E-4	133	.12759E-4	162	.29754E-5
33	.49028E-2	65	.40688E-3	74	.12918E-3	125	.37270E-4	132	.94916E-5	175	.20461E-5
65	.27539E-2	93	.28053E-3	115	.74063E-4	132	.22310E-4	158	.59479E-5	217	.62372E-6

Table 4.

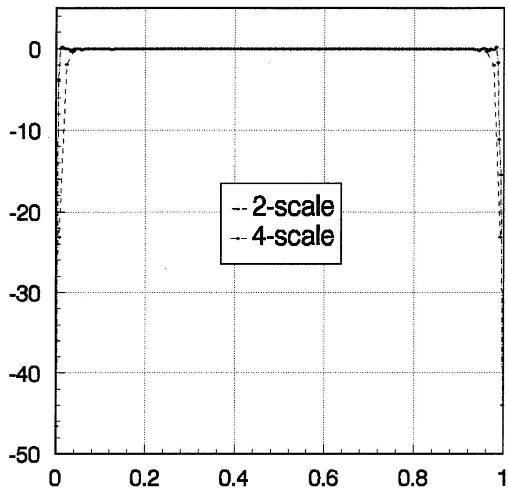


Figure 10a. The solution of the coefficients in different scales.

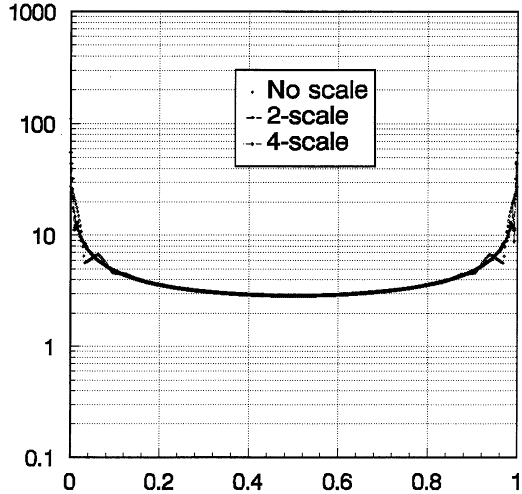


Figure 10b. The solution of the equation on different scales. Initial number of nodes $N = 32$.

6. DISCUSSIONS

For two kinds of kernel functions the solution of the Fredholm integral equation of the first kind has been presented utilizing basis functions based on different scales and different initial division of the interval.

From the numerical results, one can observe:

(1) For the multiscale moment method, it is seen that the condition number of the linear equations is poor when the number of multiscale is large. And many of the elements of the solutions of linear equations are very small.
(2) Because the condition of the linear equation will be poor as the scale increases, the scale cannot be taken too large. The solution of the linear equation has large oscillations for large scales. The adaptive multiscale moment method can improve the condition number of the linear equation resulting from the integral equation.
(3) The order of the linear equation constructed by the adaptive multiscale moment method at the i th scale will be reduced according to the solution of linear equation at the $i-1$ 'th scale
(4) The point of discontinuity of the solution can be found from the solution of the linear equation, without getting blurred too much by the Gibb's phenomenon associated with the Fourier series
(5) If the solution of the integral equation is almost a linear function, the order of the linear equation can be reduced by about 90% of the original order of the linear equation. This property shows that the method can save time in computing the numerical solution of the integral equation.
(6) The adaptive multiscale moment method can realize automatically the mesh-refinement procedure in the local regions.

Efforts in studying Fredholm integral equation of the first kind in two or three dimension by use of this new technique are under way and their results will be reported in the future.

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