Progress In Electromagnetics Research, *PIER 20, 63–73, 1998*

ANALYTICAL METHOD FOR SOLVING THE ONE-DIMENSIONAL WAVE EQUATION WITH MOVING BOUNDARY

L. Gaffour

Ecole National Superieure De Physique De Strasbourg
67400 Illkirch, France

1. INTRODUCTION

Many problems in sciences and engineering require new devices based on systems with moving boundaries. The corresponding mathematical models of these systems are initial-boundary-value-problems for the wave equation with moving boundary conditions at least at one moving boundary.

In certain simple cases [2–6], analytical solutions have been found intuitively. In general, however, more general mathematical methods are used for solving such problems [7–16]. These methods lead to very cumbersome and complex solutions. Then, it is very difficult in this case to analyze and understand correctly the physical phenomena. For this reason, the analytical method given in a previous paper [1] is generalized and used in this work. It consists in transforming the wave equation for the domain with moving boundary into a form-invariant wave equation in a domain where the boundary is fixed. The transformation is accomplished through the analogy between the Laplace's equation and the wave equation, and an analytical function permitting a conformal mapping of the two domains. The exact solutions are found in the form of a modal nature. We show that the covariant transformations of the wave equation are determined by two fundamental relations. This paper is organized as follows: Section 2 presents the formulation of the problem. Section 3 gives the solution of the problem. Section 4 treats the problem of initial conditions. Section 5 yields a necessary condition for the covariant transformation to exist. Section 6 provides some examples giving the analytical function F and the transformation functions for different cases of boundary motion

2. FORMULATION OF THE PROBLEM

Consider the hyperbolic initial-boundary-value problem with moving boundary given by the wave equation

$$U_{\tau\tau}(x,\tau) - U_{xx}(x,\tau) = \theta(x,\tau) \tag{2.1}$$

in the time-varying domain

$$\tau \geq 0, \quad 0 \leq x \leq a(\tau) \tag{2.2}$$

with the symmetric initial conditions

$$U_\tau(x,\tau)\,|_{\tau=0} = \varphi_1(x), \quad U_x(x,\tau)|_{\tau=0} = \varphi_2(x) \tag{2.3}$$

and the moving boundary conditions

$$U(0,\tau) = 0, \quad U(a(\tau),\tau) = 0 \tag{2.4}$$

where $U(x,\tau)$ is the scalar field and $\tau = ct$, c designating the characteristic wave speed.

x and t are the interdependent real space and time coordinates;
$\theta(x,\tau)$ is the exterior perturbation;
$a(\tau)$ is an arbitrary movement law of the boundary. We set $a(0) = a_0$.

$\varphi_1(x)$ and $\varphi_2(x)$ are functions giving the state of the system at $\tau = 0$. The boundary $x = 0$ is fixed.

3. SOLUTION OF THE PROBLEM

For solving this class of problems the method given in a previous paper [1] is expanded and used. Our approach, in this case, uses the analogy between Laplace's equation and the wave equation. For this we proceed to a formal change of one variable by another purely imaginary in the equation (2.1). Indeed, the change

$$\tilde{x} = ix \qquad (i^2 = -1) \tag{3.1}$$

transforms (2.1) into the following partial differential equation of the elliptic type

$$U_{\tau\tau}(\tilde{x},\tau) + U_{\tilde{x}\tilde{x}}(\tilde{x},\tau) = \theta(\tilde{x},\tau) \tag{3.2}$$

Now, it is well known that solutions of Laplace's equation remain solutions at Laplace's equation when subjected to a conformal transformation [17]. More precisely, if

$$Z = F(W) = \tilde{f}(\xi,\tilde{\eta}) + i\tilde{g}(\xi,\tilde{\eta}) \tag{3.3}$$

where F is an analytical function of the complex variable $W = \xi + i\tilde{\eta}$ with $Z = \tau + i\tilde{x}$, then the change of variables

$$\tau = \tilde{f}(\xi,\tilde{\eta}) \tag{3.4}$$

$$\tilde{x} = \tilde{g}(\xi,\tilde{\eta}) \tag{3.5}$$

transforms (2.2) into the following form

$$U_{\xi\xi}(\xi,\tilde{\eta}) + U_{\tilde{\eta}\tilde{\eta}}(\xi,\tilde{\eta}) = \theta(\xi,\tilde{\eta})\left|F'(W)\right|^2 \tag{3.6}$$

Next, we impose the following condition upon F:

$$F^*(W) = F(W^*) \tag{3.7}$$

where asterisk sign means the complex conjugate. With the help of this condition, we can show that the functions $\tilde{f}(\xi,\tilde{\eta})$ and $\tilde{g}(\xi,\tilde{\eta})$ are respectively even and odd with respect to the variable $\tilde{\eta}$. Thus, we deduce by Mac-Laurin's expansion that:

$$\tilde{f}(\xi,\tilde{\eta})\Big|_{\tilde{\eta}=i\eta} = f(\xi,\eta) \tag{3.8}$$

$$\tilde{g}(\xi,\tilde{\eta})\Big|_{\tilde{\eta}=i\eta} = ig(\xi,\eta) \tag{3.9}$$

where f and g are real functions of the real variables ξ and η. Letting $\tilde{\eta} = i\eta$ in (3.4) and (3.5) and taking (3.1), (3.8), (3.9) into account, we find the original variables τ and x as follows

$$\tau = f(\xi,\eta) \tag{3.10}$$

$$x = g(\xi,\eta) \tag{3.11}$$

With the new variables ξ and η, the equation (3.6) takes the following form:

$$U_{\xi\xi}(\xi,\eta) - U_{\eta\eta}(\xi,\eta) = \Omega(\xi,\eta) \tag{3.12}$$

where $\Omega(\xi,\eta) = \theta(\xi,\eta)\,|F'(W)|^2_{\tilde{\eta}=i\eta}$.

Thus we can conclude that the relations (3.10) and (3.11) represent the transformation functions which leave form-invariant the wave equation.

The right choice of F, produces a conformal mapping of the time-varying domain $0 \le x \le a(\tau)$ to a band $0 \le \eta \le \eta_0$. In this case, the moving boundary is transformed to a fixed boundary and the boundary conditions (2.4) could be expressed by

$$U(\xi,0) = 0, \quad U(\xi,\eta_0) = 0 \tag{3.13}$$

Now, assume that the problem is homogeneous [i.e., $\theta(x,\tau) = 0$]. Accordingly, the solution for $U(\xi,\eta)$ in the fixed domain is well-known [18, 19] and can be expressed in terms of complex Fourier series.

$$U(\xi,\eta) = \sum_{-\infty}^{\infty} A_n \left\{ \exp\frac{i\pi n}{\eta_0}(\xi+\eta) - \exp\frac{i\pi n}{\eta_0}(\xi-\eta) \right\} \tag{3.14}$$

In order to find the solution (3.14) with the original variables τ and x we consider the inverse function $\psi(Z)$ of $F(W)$ such that

$$\psi(Z) = F^{-1}(W) \tag{3.15}$$

thus we can show easily that if the condition (3.7) is satisfied, then,

$$\psi^*(Z) = \psi(Z^*) \tag{3.16}$$

Now from (3.15) and (3.16) we deduce

$$\xi + \eta = \psi(\tau + x) \tag{3.17}$$
$$\xi - \eta = \psi(\tau - x) \tag{3.18}$$

thus

$$\xi + \eta_0 = \psi\left(\tau + a(\tau)\right) \tag{3.19}$$
$$\xi - \eta_0 = \psi\left(\tau - a(\tau)\right) \tag{3.20}$$

It is obvious that the width η_0 of the band can be expressed by

$$\eta_0 = \frac{1}{2}\left[\psi\left(\tau + a(\tau)\right) - \psi\left(\tau - a(\tau)\right)\right] \tag{3.21}$$

Finally, the exact solution with the original variables is given by

$$U(x,\tau) = \sum_{-\infty}^{\infty} A_n \left\{\exp\frac{i\pi n}{\eta_0}\psi(\tau + x) - \exp\frac{i\pi n}{\eta_0}\psi(\tau - x)\right\} \tag{3.22}$$

The solution (3.22) satisfies the moving boundary conditions (2.4) Indeed, we have $U(0,\tau)=0$ and $U(a(\tau),\tau)=\sum_{-\infty}^{\infty} A_n \exp\left[\frac{i\pi n\psi(\tau - a(\tau))}{\eta_0}\right]$ $\left\{e^{i2\pi n} - 1\right\} = 0\,.$

4. PROBLEM OF THE INITIAL CONDITIONS

The determination of the coefficients A_n amounts to solving the problem of the initial conditions. We treat this question as follows:

From (3.22) we deduce

$$U_r = \sum_{-\infty}^{\infty} \frac{i\pi n}{\eta_0} A_n \left\{\psi'(\tau + x) e^{\frac{i\pi n}{\eta_0}\psi(\tau+x)} - \psi'(\tau - x) e^{\frac{i\pi n}{\eta_0}\psi(\tau-x)}\right\} \tag{4.1}$$

$$U_x = \sum_{-\infty}^{\infty} \frac{i\pi n}{\eta_0} A_n \left\{\psi'(\tau + x) e^{\frac{i\pi n}{\eta_0}\psi(\tau+x)} + \psi'(\tau - x) e^{\frac{i\pi n}{\eta_0}\psi(\tau-x)}\right\} \tag{4.2}$$

by combining (4.1) and (4.2) we obtain

$$U_x + U_\tau = \sum_{-\infty}^{\infty} \frac{2i\pi n}{\eta_0} A_n \psi'(\tau + x) e^{\frac{i\pi n}{\eta_0}\psi(\tau+x)} \tag{4.3}$$

$$U_x - U_\tau = \sum_{-\infty}^{\infty} \frac{2i\pi n}{\eta_0} A_n \psi'(\tau - x) e^{\frac{i\pi n}{\eta_0}\psi(\tau-x)} \tag{4.4}$$

and taking (2.3) into account, we obtain

$$\varphi_2(x) + \varphi_1(x) = \sum_{-\infty}^{\infty} \frac{2i\pi n}{\eta_0} A_n \psi'(x) e^{\frac{i\pi n}{\eta_0}\psi(x)} \tag{4.5}$$

$$\varphi_2(x) - \varphi_1(x) = \sum_{-\infty}^{\infty} \frac{2i\pi n}{\eta_0} A_n \psi'(-x) e^{\frac{i\pi n}{\eta_0}\psi(-x)} \tag{4.6}$$

The functions $\varphi_1(x)$ and $\varphi_2(x)$ are defined on the segment $[0, a_0]$. Let us extend $\varphi_1(x)$ odd and $\varphi_2(x)$ even on the segment $[-a_0, 0]$, then we can write

$$\varphi_1(x) = -\varphi_1(-x) \tag{4.7}$$

$$\varphi_2(x) = \varphi_2(-x) \tag{4.8}$$

This is equivalent to taking the odd continuity of $U(x,\tau)$ on the domain $[-a_0, 0]$. Thus, the expressions (4.5) and (4.6) are such that one is the consequence of the other. For convenience let us introduce the function $\kappa(x)$ such that:

$$\begin{aligned} \kappa(x) &= \varphi_2(x) + \varphi_1(x) \quad \text{if} \quad x \geq 0 \\ \kappa(x) &= \varphi_2(x) - \varphi_1(x) \quad \text{if} \quad x \leq 0 \end{aligned} \tag{4.9}$$

The functions $e^{\frac{i\pi n}{\eta_0}\psi(x)}$ are orthonormal with respect to the weight $\psi'(x)$ and also complete [17] on $[-a_0, +a_0]$. Consequently,

$$\int_{-a_0}^{+a_0} \psi'(x) \cdot e^{\frac{i\pi n}{\eta_0}(n-m)\psi(x)} dx = 2\eta_0 \delta_{n,m} \tag{4.10}$$

where $2\eta_0 = \psi(a_0) - \psi(-a_0)$ and $\delta_{n,m} = 0$ or 1 (respectively for $n \neq m$, $n = m$).

Thus, the expression of the coefficients A_n can be deduced from (4.9) and (4.10), specifically,

$$A_n = \frac{1}{4i\pi n} \int_{-a_0}^{+a_0} \kappa(x) \cdot e^{\frac{i\pi n}{\eta_0}\psi(x)} dx \tag{4.11}$$

Remarks

- The expressions (3.22) and (4.11) give the exact solution of the considered problem. This solution shows that the field $U(x, \tau)$ has a modal nature and every mode is consist of two waves in opposite directions. Because of η_0, the modes are dynamical each having an instantaneous frequency.
- The exact computation of all physical entities is possible as energy for example.
- If the problem is non homogeneous (that is when $\theta(x, \tau) \neq 0$), then the general solution is given by [18, 19]:

$$U_s = U + \frac{1}{m} \int_o^{\xi} d\xi' \int_o^{\eta} \Omega(\xi', \eta') \cdot \sin \frac{\pi n(\xi - \xi')}{\xi_0} \cdot \exp\left[\frac{\pi n(\eta - \eta')}{\eta_0}\right] d\eta' \tag{4.12}$$

where U is the solution of the homogeneous problem.

5. NECESSARY CONDITION

If the analytical function $F(W)$ satisfies the condition (3.7), then, the real transformation functions $\tau = f(\xi, \eta)$ and $x = g(\xi, \eta)$ satisfy the following fundamental relations:

$$\frac{\partial f}{\partial \xi} = \frac{\partial g}{\partial \eta} \quad \text{and} \quad \frac{\partial f}{\partial \eta} = \frac{\partial g}{\partial \xi} \tag{5.1}$$

Proof

The function $F(W) = \tilde{f}(\xi, \tilde{\eta}) + i\tilde{g}(\xi, \eta)$ is analytic; then, $\tilde{f}(\xi, \tilde{\eta})$ and $\tilde{g}(\xi, \tilde{\eta})$ satisfy the Cauchy-Riemann conditions which can be written as follows:

$$\frac{\partial \tilde{f}}{\partial \xi} = \frac{\partial \tilde{g}}{\partial \eta} \quad \text{and} \quad \frac{\partial \tilde{f}}{\partial \tilde{\eta}} = \frac{\partial \tilde{g}}{\partial \xi} \tag{5.2}$$

The substitution of $\tilde{\eta}$ by $i\eta$ yield

$$\frac{\partial f}{\partial \xi} = \frac{i\partial g}{i\partial \eta} \quad \text{and} \quad \frac{\partial f}{i\partial \eta} = \frac{i\partial g}{\partial \xi} \tag{5.3}$$

or

$$\frac{\partial f}{\partial \xi} = \frac{\partial g}{\partial \eta} \quad \text{and} \quad \frac{\partial f}{\partial \eta} = \frac{\partial g}{\partial \xi} \tag{5.4}$$

Remarks

– It is obvious that all possible transformations which leave the wave equation form-invariant have to satisfy the relations (5.1).
– These fundamental relations play the same role as the Cauchy-Riemann conditions in the transformation theory of the Laplace's equation.
– In addition to the relations (5.1), the transformation functions $\tau = f(\xi, \eta)$ and $x = g(\xi, \eta)$ are solution of wave equations.

6. APPLICATIONS

As an illustration to this work, some examples concerning the analytical function $F(W)$ and the transformation functions are given.

A-1. Linear Boundary Motion

For a domain between a fixed boundary $x = 0$ and a moving boundary $x = a(\tau)$ with $a(\tau) = a_0 + \beta\tau$ (linear boundary motion expanding $\beta > 0$ or shrinking $\beta < 0$ the x domain), the analytical function $F(W)$ permitting the conformal mapping, of the two regions is given by [1, 20]:

$$F(W) = \frac{a_0}{\beta}(e^W - 1) \tag{6.1}$$

Consequently, the transformation functions are

$$\tau = \frac{a_0}{\beta}\left(e^{\xi}\text{ch}\eta - 1\right) \tag{6.2}$$

$$x = \frac{a_0}{\beta}\text{sh}\eta \tag{6.3}$$

and the function $\psi(Z) = f^{-1}(W)$ is given by:

$$\psi(Z) = W = \ln\left(1 + \frac{a_0}{\beta}Z\right) \tag{6.4}$$

Thus, the width η_0 of the band is

$$\eta_0 = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \tag{6.5}$$

A-2. Parabolic Boundary Motion

If the boundary moves with the law $a(\tau) = \sqrt{a_0^2 + 2\beta\tau}$ (parabolic boundary motion), then we obtain

$$F(W) = \frac{a_0\beta}{2}\left[\left(W + \frac{1}{\beta^2}\right)^2 - 1\right] \tag{6.6}$$

$$\tau = \frac{a_0\beta}{2}\left[\eta^2 + \left(\xi + \frac{1}{\beta^2}\right)^2 - 1\right] \tag{6.7}$$

$$x = a_0\beta\eta\left(\xi + \frac{1}{\beta^2}\right) \tag{6.8}$$

A-3. Elliptic Boundary Motion

If $a(\tau) = \sqrt{a_0^2 + \beta^2\tau^2}$ (elliptic boundary motion), then:

$$F(W) = \frac{a_0}{\beta}\sqrt{1-\beta^2} \cdot sh(W) \tag{6.9}$$

$$\tau = \frac{a_0}{\beta}\sqrt{1-\beta^2} \cdot sh\xi \cdot ch\eta \tag{6.10}$$

$$x = \frac{a_0}{\beta}\sqrt{1-\beta^2} \cdot ch\xi \cdot sh\eta \tag{6.11}$$

A-4. Remark

It is remarkable that the conditions (3.7) and (3.15) are satisfied respectively by $F(W)$ and $\Psi(Z)$, and also, the fundamental relations (5.1) are satisfied by the expressions of τ and x.

B. The Lorentz Transformation

The fundamental relations (5.1) determine all covariant transformations of the wave equation. These transformations are, in general, non-linear. However, it is possible to find linear transformations which leave the wave equation form-invariant. As example, we give the Lorentz transformation between inertial frames (τ, x) and (ξ, η):

$$\tau = \gamma(\eta + \beta\xi), \quad x = \gamma(\xi + \beta\eta) \tag{6.12}$$

where

$$\beta = \frac{v}{c}, \quad \gamma = \sqrt{1-\beta^2} \tag{6.13}$$

We can show easily that the linear Lorentz transformation satisfies the conditions (5.1) and are solution of wave equations. Indeed,

$$\tau_\xi = x_\eta = \beta\gamma, \quad \tau_\eta = x_\xi = \gamma \tag{6.14}$$

and

$$\tau_{\xi\xi} = \tau_{\eta\eta} = 0, \quad x_{\xi\xi} = x_{\eta\eta} = 0 \tag{6.15}$$

We can conclude that the linear Lorentz transformation is only a simple transformation of a certain class of transformations which are, in general, non-linear.

7. CONCLUSION

The analytical method given in a previous paper [1] is generalized for studying a large class of dynamic systems with moving boundaries. On one hand, the method is general and can be applied whatever the movement of the boundary may be; on the other hand, the solutions given by this method are exact, the analysis of the physical phenomena is possible and all needed physical entities can be computed. This method is simple and efficient compared to others existing in the literature. The study, by this method, of the electromagnetic field in a plane resonator with moving boundary will be given in another paper.

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