Progress In Electromagnetics Research, *PIER 23, 301–314, 1999*

EFFECTIVE BOUNDARY CONDITIONS FOR A 2D INHOMOGENEOUS NONLINEAR THIN LAYER COATED ON A METALLIC SURFACE

M. Norgren and S. He

Department of Electromagnetic Theory
Royal Institute of Technology
S-100 44 Stockholm, Sweden

1. INTRODUCTION

Approximate boundary conditions have been widely used in problems of wave propagation, radiation and guidance to simulate the material and geometric properties of surfaces. For example, a metal plane which is coated by a thin linear dielectric layer can be replaced by an effective impedance boundary condition [1–4]. The general purpose of the approximate effective boundary conditions is to simplify the analytical or numerical solution of wave scattering problem involving complex structures by e.g., converting a multiple-medium problem into a single medium problem with a simple smooth boundary surface. In the present paper we derive some effective boundary conditions for a two-dimensional inhomogeneous nonlinear thin layer coated on a perfectly conducting surface. Nonlinear inhomogeneous thin films have

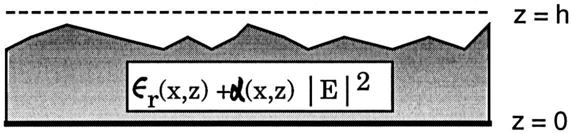

Figure 1. The geometry of the problem.

many important applications in e.g., self-focusing, second harmonic generation, optical amplification, couplers. In nonlinear guided optical phenomena, a Kerr-type nonlinear medium whose refractive index depends on light intensity is often used (see e.g., [5–6]). A thin coating of such a Kerr-type nonlinear medium will be considered in the present paper. The simple case of a thin layer coated on a planar metallic surface is considered first. An asymptotic expansion of the field solution in power series of the thickness is used after a suitable scaling along the vertical direction with the thickness of the thin layer, and a second order approximate boundary condition is derived. Numerical results show that the second order approximate boundary condition gives a sufficient accuracy for all incident angles when the coating thickness is much smaller than the wavelength. The results are generalized to the case when the nonlinear thin layer is coated on a curved metallic surface.

2. AN INHOMOGENEOUS NONLINEAR THIN COATING ON A PLANAR METALLIC SURFACE

In this section we consider a thin layer of inhomogeneous nonlinear medium situated in $0 \le z \le h$ (the $+z$ direction points upward; h is the thickness of the thin layer; see Fig. 1 for the geometry of the problem). The inhomogeneous thin layer is superimposed on a perfectly conducting surface at $z = 0$. We consider a Kerr-type of nonlinear medium, which is often used in nonlinear guided optical phenomena (see e.g., [5–6]). Thus inside the thin layer the relative permittivity is $\epsilon_r + \alpha|\mathbf{E}|^2$. The permittivity is assumed to be y-independent. Above the nonlinear thin layer there is a vacuum with a permittivity ϵ_0. The permeability has a constant value μ_0 everywhere.

For simplicity we consider the TE case for which the electric field is perpendicular to the xz plane (i.e., the electric field is along the y direction) and the field is y-independent. The time-dependence of all fields is assumed to be $e^{-i\omega t}$. Then the electric field $\mathbf{E} = E(x,z)\mathbf{i}_y$, where $\mathbf{i}_y$ is the unit vector along the y direction, and the amplitude $E(x,z)$ satisfies the following nonlinear differential equation

$$(\partial_x^2 + \partial_z^2)E + k_0^2 \left[\epsilon_r(x,z) + \alpha(x,z)|E|^2\right] E = 0, \tag{1}$$

where $\partial_z = \frac{\partial}{\partial z}$ and the wave number is

$$k_0 = \omega\sqrt{\mu_0\epsilon_0}. \tag{2}$$

The boundary condition at $z = 0$ is

$$E|_{z=0} = 0. \tag{3}$$

From the continuity of the tangential components of electric and magnetic fields it follows that both E and $\partial_z E$ are continuous cross any z plane (note that $\partial_z E = -i\omega\mu_0 H_x$, where H_x is the x component of the magnetic field).

Introduce a scaled coordinate system (x,τ) with the variable τ defined by

$$\tau = z/h. \tag{4}$$

Let $\epsilon_r(x,\tau)$, $\alpha(x,\tau)$ denote the values of ϵ_r and α at the point (x,τ) in the scaled coordinate system (corresponding to the point $(x,\tau h)$ in the original (x,z) coordinate system). We want to describe the asymptotic behavior of $E(x,z;h)$ inside the nonlinear thin layer as h goes to zero under the assumption that the "scaled" profiles $\epsilon_r(x,\tau)$ and $\alpha(x,\tau)$ remain unchanged. More precisely, we want to derive an approximate boundary condition on the surface $z = h$. We can find such a condition by a scale analysis as follows. Introduce an asymptotic expansion of $E(x,z;h)$ of the following form:

$$\begin{aligned} E(x,z;h) = E^{(0)}(x,\tau) &+ hE^{(1)}(x,\tau) \\ &+ h^2E^{(2)}(x,\tau) + h^3E^{(3)}(x,\tau) + \dots, \end{aligned} \tag{5}$$

for $z \in [0,h]$. Then, in the scaled coordinate system (x,τ), the differential equation (1) becomes

$$\left(\partial_x^2 + \frac{1}{h^2}\partial_\tau^2\right) E + k_0^2\epsilon_r(x,\tau)E + k_0^2\alpha(x,\tau)|E|^2E = 0. \tag{6}$$

The approximate boundary condition that we are looking for is a relation between $\partial_z E(x,z;h)$ and $E(x,z;h)$ at the surface $z = h$.

We substitute the expansion (5) into Eq. (6), and match the coefficients of the $h^{-2}, h^{-1}, h^0, h, h^2, \ldots,$ terms, respectively. In doing this, we assume that the thin layer does not vary very rapidly in the lateral direction, so that we can treat $\partial_x E$ as a quantity of order $O(1)$ (otherwise one has to combine e.g., the homogenization theory to average the material variation in the lateral direction; see e.g., [7]). Thus we obtain

$$\partial_\tau^2 E^{(0)} = 0, \tag{7}$$

$$\partial_\tau^2 E^{(1)} = 0, \tag{8}$$

$$\partial_\tau^2 E^{(2)} + \partial_x^2 E^{(0)} + k_0^2 \epsilon_r(x,\tau) E^{(0)} + k_0^2 \alpha(x,\tau) |E^{(0)}|^2 E^{(0)} = 0, \tag{9}$$

$$\begin{aligned} &\partial_\tau^2 E^{(3)} + \partial_x^2 E^{(1)} + k_0^2 \epsilon_r(x,\tau) E^{(1)} \\ &\qquad + k_0^2 \alpha(x,\tau) \left[2 \left| E^{(0)} \right|^2 E^{(1)} + \left(E^{(0)} \right)^2 \overline{E}^{(1)} \right] = 0, \end{aligned} \tag{10}$$

where $\overline{E}$ denotes the complex conjugate of E.

The boundary condition (3) becomes

$$E^{(0)}(x,0) + hE^{(1)}(x,0) + h^2 E^{(2)}(x,0) + \ldots = 0,$$

which gives

$$E^{(p)}(x,0) = 0, \qquad p = 0, 1, 2, 3, \ldots \tag{11}$$

It then follows from Eqs. (7) and (8) that

$$\partial_\tau E^{(0)}(x,\tau) = C_0(x), \tag{12}$$

$$\partial_\tau E^{(1)}(x,\tau) = C_1(x), \tag{13}$$

$$E^{(0)}(x,\tau) = C_0(x)\,\tau, \tag{14}$$

$$E^{(1)}(x,\tau) = C_1(x)\,\tau, \tag{15}$$

where $C_0(x)$ and $C_1(x)$ are functions depending only on x.

From Eqs. (9) and (14), one obtains

$$\partial_\tau^2 E^{(2)} = -\tau \partial_x^2 C_0(x) - k_0^2 \tau \epsilon_r(x,\tau) C_0(x) - k_0^2 \alpha(x,\tau) |C_0(x)|^2 C_0(x) \tau^3. \tag{16}$$

Integrating the above equation with respect to τ , yields

$$\partial_\tau E^{(2)} = -\frac{1}{2}\tau^2\partial_x^2 C_0(x) - k_0^2 C_0(x)\int_0^\tau \tau_1 \epsilon_r(x,\tau_1)\,d\tau_1 - k_0^2|C_0(x)|^2 C_0(x)\int_0^\tau \tau_1^3\alpha(x,\tau_1)\,d\tau_1 + C(x), \tag{17}$$

$$E^{(2)} = -\frac{1}{6}\tau^3\partial_x^2 C_0(x) - k_0^2 C_0(x)\int_0^\tau \left[\int_0^{\tau_1} \tau_2\epsilon_r(x,\tau_2)\,d\tau_2\right] d\tau_1 - k_0^2|C_0(x)|^2 C_0(x)\int_0^\tau \left[\int_0^{\tau_1} \tau_2^3\alpha(x,\tau_2)\,d\tau_2\right] d\tau_1 + C(x)\tau, \tag{18}$$

where $C(x)$ is a function depending only on x . Putting $\tau = 1$ in Eq. (18), one obtains

$$C(x) = E^{(2)}(x,1) + \frac{1}{6}\partial_x^2 C_0(x) + k_0^2 C_0(x)\int_0^1 \left[\int_0^{\tau_1} \tau_2\epsilon_r(x,\tau_2)\,d\tau_2\right] d\tau_1 + k_0^2|C_0(x)|^2 C_0(x)\int_0^1 \left[\int_0^{\tau_1} \tau_2^3\alpha(x,\tau_2)\,d\tau_2\right] d\tau_1. \tag{19}$$

Substituting Eq. (19) into Eq. (17) with $\tau = 1$, yields

$$\left(\partial_\tau E^{(2)}\right)(x,1) = E^{(2)}(x,1) - \frac{1}{3}\partial_x^2 C_0(x) - \frac{1}{3}k_0^2 C_0(x)\tilde{\epsilon}_r(x) - \frac{1}{5}k_0^2|C_0(x)|^2 C_0(x)\tilde{\alpha}(x). \tag{20}$$

where

$$\tilde{\epsilon}_r(x) = 3\left\{\int_0^1 \tau_1\epsilon_r(x,\tau_1)\,d\tau_1 - \int_0^1 \left[\int_0^{\tau_1} \tau_2\epsilon_r(x,\tau_2)\,d\tau_2\right] d\tau_1\right\}, \tag{21}$$

$$\tilde{\alpha}(x) = 5\left\{\int_0^1 \tau_1^3\alpha(x,\tau_1)\,d\tau_1 - \int_0^1 \left[\int_0^{\tau_1} \tau_2^3\alpha(x,\tau_2)\,d\tau_2\right] d\tau_1\right\}. \tag{22}$$

In the special case when the nonlinear thin layer has only lateral variation, i.e., $\epsilon_r(x,\tau) = \epsilon_r(x)$, $\alpha(x,\tau) = \alpha(x)$, one has $\tilde{\epsilon}_r(x) = \epsilon_r(x)$ and $\tilde{\alpha}(x) = \alpha(x)$.

Therefore, one obtains from Eqs. (12)–(15) and (20) that

$$\begin{aligned}
\left(\partial_\tau E\right)(x,1) &= \left(\partial_\tau E^{(0)}\right)(x,1) + h\left(\partial_\tau E^{(0)}\right)(x,1) \\
&\quad + h^2\left(\partial_\tau E^{(0)}\right)(x,1) + O(h^3) \\
&= C_0(x) + hC_1(x) + h^2 E^{(2)}(x,1) - \frac{1}{3}h^2\partial_x^2 C_0(x) \\
&\quad - \frac{1}{3}h^2 k_0^2 C_0(x)\tilde{\epsilon}_r(x) - \frac{1}{5}h^2 k_0^2 |C_0(x)|^2 C_0(x)\tilde{\alpha}(x) \\
&\quad + O(h^3) \\
&= E^{(0)}(x,1) + hE^{(1)}(x,1) + h^2 E^{(2)}(x,1) \\
&\quad - \frac{1}{3}h^2\partial_x^2 E^{(0)}(x,1) - \frac{1}{3}h^2 k_0^2 E^{(0)}(x,1)\tilde{\epsilon}_r(x) \\
&\quad - \frac{1}{5}h^2 k_0^2 |E^{(0)}(x,1)|^2 E^{(0)}(x,1)\tilde{\alpha}(x) + O(h^3) \\
&= \left[E(x,1) + O(h^3)\right] - \frac{1}{3}h^2\partial_x^2\left[E(x,1) + O(h)\right] \\
&\quad - \frac{1}{3}h^2 k_0^2\tilde{\epsilon}_r(x)\left[E(x,1) + O(h)\right] \\
&\quad - \frac{1}{5}h^2 k_0^2\tilde{\alpha}(x)\left|\left[E(x,1) + O(h)\right]\right|^2\left[E(x,1) + O(h)\right] \\
&\quad + O(h^3) \\
&= E(x,1) - \frac{1}{3}h^2\partial_x^2 E(x,1) - \frac{1}{3}h^2 k_0^2\tilde{\epsilon}_r(x)E(x,1) \\
&\quad - \frac{1}{5}h^2 k_0^2\tilde{\alpha}(x)|E(x,1)|^2 E(x,1) + O(h^3),
\end{aligned}$$

which gives the following second order approximation in the scaled coordinate system

$$\begin{aligned}
\left(\partial_\tau E\right)(x,1) &= E(x,1) - \frac{1}{3}h^2\partial_x^2 E(x,1) - \frac{1}{3}h^2 k_0^2\tilde{\epsilon}_r(x)E(x,1) \\
&\quad - \frac{1}{5}h^2 k_0^2\tilde{\alpha}(x)|E(x,1)|^2 E(x,1).
\end{aligned} \tag{23}$$

In the physical (x,z) coordinate system, the above equation becomes the following second order approximate boundary condition

$$\begin{aligned}
&E(x,z) - h\partial_z E(x,z) - \frac{1}{3}h^2\partial_x^2 E(x,z) - \frac{1}{3}h^2 k_0^2\tilde{\epsilon}_r(x)E(x,z) \\
&\quad - \frac{1}{5}h^2 k_0^2\tilde{\alpha}(x)|E(x,z)|^2 E(x,z), \qquad z = h.
\end{aligned} \tag{24}$$

Therefore, one can replace the inhomogeneous nonlinear thin structure with the above boundary condition in e.g. the computation of the scattered field above the nonlinear thin layer. Note that the boundary condition (24) is still valid when the thin coating layer is vacuum, i.e., $\epsilon_r = 1$ and $\alpha = 0$.

In a similar way one can derive also higher order approximate boundary conditions for the inhomogeneous nonlinear thin coating.

A numerical example.

As a numerical example, we consider a stratified nonlinear thin coating with the following profiles

$$\begin{aligned} \epsilon_r(z) &= \left[4 + 2\cos\frac{3\pi z}{h}\right], \\ \alpha(z) &= \frac{1}{2}\left[1 + \sin\frac{3\pi z}{2h}\right], \qquad 0 < z < h \end{aligned}$$

Equations (21) and (22) then gives

$$\tilde{\epsilon}_r = 3.864905087, \qquad \tilde{\alpha} = 0.1971738964.$$

The thickness of the coating is chosen to be $h = 0.1\pi/k_0$. We consider a plane wave obliquely incident (with the incident angle θ) on this stratified nonlinear thin layer. As shown in the Appendix, if the incident electric field on the surface is chosen to be

$$E^{inc} = 1, \qquad z = h,$$

then the reflected electric field can be described by a (real-valued) phase ϕ, i.e.,

$$E^{refl} = e^{i\phi}, \qquad z = h,$$

when both ϵ_r and α are real-valued. The dashed curve in Fig. 2 gives the reflection phase ϕ as a function of the incident angle θ when the second order approximate boundary condition (24) is applied (see the appendix for the numerical algorithm). The solid curve gives the true values for ϕ calculated with a method for scattering from a stratified nonlinear slab, namely, the shooting method [8]. The dotted curve in Fig. 2 gives the corresponding values when the nonlinear thin coating is replaced by a vacuum. The numerical results in Fig. 2 show

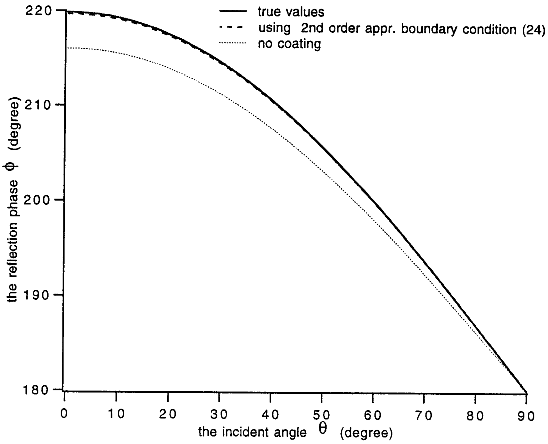

Figure 2. The reflection phase ϕ as a function of the incident angle θ. The dashed curve is obtained by using the effective boundary condition (24). The solid curve is for the true values calculated with a shooting method.

that the second order approximate boundary condition gives scattering results of good accuracy for all incident angles when the thickness of the coating is much smaller than the wavelength.

3. AN INHOMOGENEOUS NONLINEAR THIN COATING ON A CURVED METALLIC SURFACE

In this section we generalize the results derived in the previous section to the case when the inhomogeneous nonlinear thin layer is coated on a metallic cylinder of an arbitrary smooth cross section described by its boundary curve Γ.

Outside but sufficiently close to Γ, we denote by $\mathbf{r}_\Gamma$ the orthogonal projection of a point $\mathbf{r}$ on Γ, s a curvilinear abscissa (tangential coordinate) of $\mathbf{r}_\Gamma$, and

$$n = |\mathbf{r} - \mathbf{r}_\Gamma|. \tag{25}$$

Then (s, n) is a parameterization of the neighborhood of the curve Γ. The unit normal to the curve Γ at $\mathbf{r}_\Gamma$ is denoted by $\hat{\mathbf{n}}$. Denote by $c(s, n)$ the curvature at the point (s, n) of the curve

$$\Gamma_n \equiv \{\mathbf{r} = \mathbf{r}_\Gamma + n\hat{\mathbf{n}}\}, \tag{26}$$

which is "parallel" to the curve Γ. In a special case when the metallic object is a circular cylinder with radius a, one has

$$c(s,n) = \frac{1}{a+n}.$$

The length element ds_n on the curve Γ_n at the point $\mathbf{r}$ is related to the length element ds on the curve Γ at the point $\mathbf{r}_\Gamma$ by

$$ds_n = [1 + c(s,0)n]ds.$$

Thus one has

$$\partial_{s_n} = \frac{1}{1 + c(s,0)n}\,\partial_s. \tag{27}$$

The Laplacian Δ has the following form in the local coordinate system (s,n) when the field has no variation along the axis of the cylinder (see e.g. [9]–[10]):

$$\Delta = \partial_n^2 + c(s,n)\partial_n + \partial_{s_n}^2. \tag{28}$$

Now consider a two-dimensional perfectly conducting object (with boundary Γ) coated with an inhomogeneous nonlinear thin layer of a thickness h. Outside the nonlinear thin layer there is a vacuum. The electric field is assumed to be parallel to the axis of the cylinder. Then the amplitude $E(s,n)$ of the electric field satisfies the following nonlinear differential equation (cf. Eq.(1))

$$\Delta E + k_0^2\left[\epsilon_r(s,n) + \alpha(s,n)|E|^2\right]E = 0. \tag{29}$$

The boundary condition at $n = 0$ is

$$E = 0, \qquad\qquad n = 0. \tag{30}$$

Let

$$\tau = n/h, \tag{31}$$

and introduce the following asymptotic expansion:

$$E(s,n;h) = E^{(0)}(s,\tau) + hE^{(1)}(s,\tau) + h^2E^{(2)}(s,\tau) + \ldots. \tag{32}$$

For small h, the curvature $c(s,n)$ can be expanded as

$$c(s,h\tau) = c(s,0) + h\tau c'(s,0) + \ldots, \tag{33}$$

where $c'(s,0) = [\partial_n c(s,n)]_{n=0}$. The tangential derivative in the expression (28) has the following Taylor$'$s expansion (cf. Eq. (27)):

$$\partial^2_{s_n} = \partial^2_s - \left[2c(s,0)\tau\partial^2_s + \partial_s c(s,0)\tau\partial_s\right] h + \dots.$$

Thus, substituting the expansion (32) into the differential equation (29), and matching the coefficients of the $h^{-2}, h^{-1}, h^0, \dots,$ terms, respectively, one obtains

$$\partial^2_\tau E^{(0)} = 0, \tag{34}$$

$$\partial^2_\tau E^{(1)} + c(s,0)\partial_\tau E^{(0)} = 0, \tag{35}$$

$$\begin{aligned}\partial^2_\tau E^{(2)} + \partial^2_s E^{(0)} + c(s,0)\partial_\tau E^{(1)} + \tau c'(s,0)\partial_\tau E^{(0)} \\ + k_0^2\epsilon_r(s,\tau)E^{(0)} + k_0^2\alpha(s,\tau)|E^{(0)}|^2 E^{(0)} = 0,\end{aligned} \tag{36}$$

$$\dots$$

The boundary condition (30) becomes

$$E^{(p)}(s,0) = 0, \qquad\qquad p = 0,1,2,\dots \tag{37}$$

It then follows from Eqs. (34) and (35) that

$$E^{(0)}(s,\tau) = C_0(s)\,\tau, \tag{38}$$

$$E^{(1)}(s,\tau) = -\frac{\tau^2}{2}c(s,0)C_0(s) + C_1(s)\,\tau, \tag{39}$$

where $C_0(s)$ and $C_1(s)$ are certain functions depending only on the tangential coordinate s.

Substituting Eqs. (38) and (39) into Eq. (36), yields

$$\begin{aligned}\partial^2_\tau E^{(2)} =& - k_0^2\tau\epsilon_r(s,\tau)C_0 - \tau[\partial^2_s C_0(s) - c^2(s,0)C_0(s) + c'(s,0)C_0(s)] \\ & - c(s,0)C_1(s) - k_0^2\alpha(s,\tau)|C_0(s)|^2 C_0(s)\tau^3.\end{aligned} \tag{40}$$

Solving the above differential equation in a way similar to the one that leads to Eq. (20), one obtains

$$\begin{aligned}\left(\partial_\tau E^{(2)}\right)(s,1) = E^{(2)}(s,1) - \frac{1}{3}\omega^2\mu_1 C_0(s)\tilde{\epsilon}(s) - \frac{1}{3}[\partial^2_s C_0(s) \\ - c^2(s,0)C_0(s) + c'(s,0)C_0(s)] \\ - \frac{1}{2}c(s,0)C_1(s) - \frac{1}{5}h^2k_0^2\tilde{\alpha}(s)|E(s,1)|^2 E(s,1).\end{aligned} \tag{41}$$

where

$$\tilde{\epsilon}(s) = 3\left\{\int_0^1 \tau_1 \epsilon(s,\tau_1)\,d\tau_1 - \int_0^1 \left[\int_0^{\tau_1} \tau_2 \epsilon(s,\tau_2)\,d\tau_2\right] d\tau_1\right\}, \tag{42}$$

$$\tilde{\alpha}(s) = 5\left\{\int_0^1 \tau_1^3 \alpha(s,\tau_1)\,d\tau_1 - \int_0^1 \left[\int_0^{\tau_1} \tau_2^3 \alpha(s,\tau_2)\,d\tau_2\right] d\tau_1\right\}. \tag{43}$$

Combining Eqs. (38), (39) and (41), and following a derivation similar to the one that leads to Eq. (24), one can obtains the following second order approximate boundary condition for the inhomogeneous nonlinear thin coating:

$$\begin{aligned} &E - h\,\partial_n E - \frac{1}{2}hc(s,0)\,E - \frac{1}{3}h^2 k_0^2 \tilde{\epsilon}_r(s)\,E \\ &- \frac{1}{3}h^2\left[\partial_s^2 E - \frac{1}{4}c^2(s,0)\,E + c'(s,0)E\right] - \frac{1}{5}h^2 k_0^2 \tilde{\alpha}(s)|E|^2 E, \\ &\qquad n = h. \end{aligned} \tag{44}$$

4. CONCLUSION

An effective boundary condition for an inhomogeneous nonlinear thin layer coated on a metallic plane has been derived through an asymptotic expansion of the field in power series of the thickness. Numerical results have shown that the effective boundary condition gives good accuracy for all incident angles when the coating thickness is much smaller than the wavelength. The results have been generalized to the case when the inhomogeneous nonlinear thin layer is coated on a curved metallic surface. The boundary conditions derived in the present paper can be used effectively to replace an inhomogeneous nonlinear thin layer coated on a metallic surface. The results can be extended to the three-dimensional case in a way similar to the one described in [4].

APPENDIX. A NUMERICAL SOLUTION TO A SCATTERING PROBLEM WITH THE APPROXIMATE BOUNDARY CONDITION (24)

In this appendix we describe a numerical solution to a scattering problem with the approximate boundary condition (24) when both $\tilde{\epsilon}_r$ and

$\tilde{\alpha}$ are real-valued constants. For an obliquely incident plane wave with the incidence angle θ, the boundary condition becomes

$$E - h\partial_z E - \frac{1}{3}h^2k_0^2(\tilde{\epsilon}_r - \sin^2\theta)E - \frac{1}{5}h^2k_0^2\tilde{\alpha}|E|^2E = 0, \quad z = h. \quad \text{(A1)}$$

At the surface $z = h$ (in the vacuum), the incident and reflected electric fields have the following expressions [11–12]:

$$E^{inc} = \frac{1}{2}\left(E + \sqrt{\mu_0/\epsilon_0}H_x/\cos\theta\right),$$
$$E^{refl} = \frac{1}{2}\left(E - \sqrt{\mu_0/\epsilon_0}H_x/\cos\theta\right),$$

where H_x is the x component of the magnetic field. Thus, one obtains

$$E = E^{inc} + E^{refl}, \quad \text{(A2)}$$
$$\begin{aligned}\partial_z E &= -i\omega\mu_0 H_x = -i\omega\mu_0\cos\theta\sqrt{\epsilon_0/\mu_0}\left(E^{inc} - E^{refl}\right)\\ &= -ik_0\cos\theta\left(E^{inc} - E^{refl}\right). \quad \text{(A3)}\end{aligned}$$

Substituting Eqs. (A2) and (A3) into Eq. (A1), one obtains

$$KE^{inc} + \bar{K}E^{refl} = 0, \quad \text{(A4)}$$

where $\overline{K}$ is the complex conjugate of K, and

$$K = 1 - \frac{1}{3}h^2k_0^2(\tilde{\epsilon}_r - \sin^2\theta) - \frac{1}{5}h^2k_0^2\tilde{\alpha}|E|^2 + ihk_0\cos\theta.$$

Equation (A4) indicates that

$$\left|\frac{E^{refl}}{E^{inc}}\right| = 1. \quad \text{(A5)}$$

As in the numerical example given in Section 2, we choose $E^{inc} = 1$ at $z = h$. Thus $|E^{refl}| = 1$ and the reflection is described by a phase, i.e.,

$$E^{inc} = 1, \quad \text{(A6)}$$
$$E^{refl} = e^{i\phi}, \quad \text{(A7)}$$

where the phase ϕ is real-valued. Substituting Eqs. (A6) and (A7) into Eq. (A1), one obtains the following equation for ϕ,

$$A\cos\frac{\phi}{2} - 4B\cos^3\frac{\phi}{2} + C\sin\frac{\phi}{2} = 0, \tag{A8}$$

where

$$A = 1 - \frac{1}{3}h^2k_0^2(\tilde{\epsilon}_r - \sin^2\theta), \tag{A9}$$

$$B = \frac{1}{5}h^2k_0^2\tilde{\alpha}, \tag{A10}$$

$$C = hk_0\cos\theta. \tag{A11}$$

Equation (A8) can be written as

$$16B^2\left[\cos^2\frac{\phi}{2}\right]^3 - 8AB\left[\cos^2\frac{\phi}{2}\right]^2 + (A^2+C^2)\left[\cos^2\frac{\phi}{2}\right] - C^2 = 0. \tag{A12}$$

The above cubic algebraic equation for $\cos^2\frac{\phi}{2}$ must have at least one non-negative real root since the product of the three roots equals $\frac{C^2}{16B^2}$ (non-negative). If x_0 denotes such a non-negative root in the region $[0,1]$, then

$$\cos\frac{\phi}{2} = \pm\sqrt{x_0}. \tag{A13}$$

Choose $\phi \in [0, 2\pi]$, then $\sin\frac{\phi}{2} \geq 0$. Since (cf. Eq. (A8))

$$\sin\frac{\phi}{2} = -\frac{1}{C}\cos\frac{\phi}{2}\left(A - 4B^2\cos^2\frac{\phi}{2}\right), \tag{A14}$$

the sign in Eq. (A13) is determined by the fact that $\sin\frac{\phi}{2}$ is non-negative. Therefore, one uniquely determines the numerical value of the phase $\phi \in [0, 2\pi]$ for a fixed incident angle θ.

ACKNOWLEDGMENT

The partial support of the Swedish Research Council for Engineering Sciences is gratefully acknowledged.

REFERENCES

1. Harrington, R. F., and J. R. Mautz, "An impedance sheet conditions for thin dielectric shells," *IEEE Trans. Antennas Propagat.,* Vol. AP-23, 531–534, 1975.
2. Bruno, O. P., and F. Reitich, "Solution of a boundary value problem for the Hemholtz equation via variation of the boundary into the complex domain," *Proc. Royal Soc. Edinburgh,* 317–340, 1993.
3. Senior, T. B. A., and J. L. Volakis, *Approximate Boundary Conditions in Electromagnetics,* IEE, UK, 1995.
4. Ammari, H., and S. He, "Effective impedance boundary conditions for an inhomogeneous thin layer on a curved metallic surface," *IEEE Trans. Antennas Propagat.,* Vol. 46, No. 5, 710–715, 1998.
5. Winful, H. G., J. H. Marburger, and E. Garmire, "Theory of bistability in nonlinear distributed feedback structure," *Appl. Phys. Lett.,* Vol. 35, 379–381, 1979.
6. Yokota, M., "Analysis of nonlinear grating couplers by singular perturbation technique," *J. Lightwave Technol.,* Vol. 12, No. 12, 2049–2055, 1994.
7. Ammari, H., and S. He, "Homogenization and scattering for gratings," *J. Electromagn. Waves Appl.,* Vol. 11, 1669–1683, 1997.
8. Press, W. H., B. P. Flannery, A. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes – The Art of Scientific Computing,* Chapter 16, Cambridge University Press, Cambridge, 1986.
9. Do Carmo, M. P., *Differential Geometry of Curves and Surfaces,* Pretince Hall, Englewood Cliffs, New Jersey, 1976.
10. Abboud, T., and H. Ammari, "Diffraction at a curved grating: TM and TE cases, Homogenization," *J. Math. Anal. Appl.,* Vol. 202, 995–1026, 1996.
11. He, S., S. Ström, and V. H. Weston, *Time Domain Wave-splitting and Inverse Problems,* Oxford University Press, Oxford, 1998.
12. Norgren, M., and S. He, "General scheme for electromagnetic reflection and transmission for composite structures of complex materials," *Proc. IEE, Part H: Microwaves, Antennas and Propagation,* Vol. 142, No. 1, 52–56, 1995.