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FRACTIONAL DUAL SOLUTIONS AND CORRESPONDING SOURCES

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1. INTRODUCTION

Fractional calculus [1] is an area of mathematics that deals with operators having noninteger orders. In recent years, Engheta [2–6] has been interested in exploring the roles and applications of fractional calculus in electromagnetics. He applied the concept of fractional derivatives/integrals in several electromagnetics problems and obtained results which demonstrate that these fractional operators can be useful tools in electromagnetics.

In a recent work, Engheta [7] has extended the idea of fractionalizing the differential and integral operators by introducing the concept of fractional curl operator, i.e., curl^α. In the new fractional operator, when $\alpha = 1$, conventional curl operator is obtained and for $\alpha = 0$, identity operator, meaning no operation, is obtained. For α other than zero and one, curl^α can be regarded as fractional curl operator.

It may be noted that curl operator in Fourier transform domain [from (x, y, z)-domain to $\mathbf{k}$-domain i.e., (k_x, k_y, k_z)] is equivalent to cross product operator, i.e., $(i\mathbf{k}\times)$.

In electromagnetics, **principle of duality**, states that if ($\mathbf{E}$, $\mathbf{H}$) is the one set of field solutions satisfying the Maxwell's equations then the other set of field solutions satisfying the Maxwell's equations will be $(\eta\mathbf{H}, -\mathbf{E}/\eta)$. $\eta = \sqrt{\mu/\epsilon}$ is the impedance of the medium. Solution set ($\mathbf{E}$, $\mathbf{H}$) is termed as original solutions to Maxwell's equations while solution set $(\eta\mathbf{H}, -\mathbf{E}/\eta)$ is termed as dual solutions to Maxwell's equations. Engheta utilized the idea of fractional curl operator to fractionalize the principle of duality by introducing the new set of field solutions for the source-free Maxwell's equations. These solutions can be regarded as fractional solutions between the original and dual solutions to the Maxwell's equations. For a given original set of field solutions, i.e., ($\mathbf{E}$, $\mathbf{H}$), with $\mathbf{k}$ as direction of propagation, fractional dual solutions can be obtained as [7]

$$\widetilde{\mathbf{E}}_{fd} = \left[\frac{1}{(ik)^\alpha}(i\mathbf{k}\times)^\alpha \widetilde{\mathbf{E}}\right]$$
$$\eta\widetilde{\mathbf{H}}_{fd} = \left[\frac{1}{(ik)^\alpha}(i\mathbf{k}\times)^\alpha (\eta\widetilde{\mathbf{H}})\right]$$

where $k = |\mathbf{k}| = \omega\sqrt{\mu\epsilon}$. Symbol $\sim$ is used to represent that the quantity is in $\mathbf{k}$-domain. Corresponding expressions in the (x, y, z)-domain can be written as

$$\mathbf{E}_{fd} = \left[\frac{1}{(ik)^\alpha}\text{curl}^\alpha \mathbf{E}\right]$$
$$\eta\mathbf{H}_{fd} = \left[\frac{1}{(ik)^\alpha}\text{curl}^\alpha (\eta\mathbf{H})\right].$$

For $\alpha = 0$, above expressions will yield $\mathbf{E}_{fd} = \mathbf{E}$ and $\eta\mathbf{H}_{fd} = \eta\mathbf{H}$, which are original field solutions satisfying the Maxwell's equations, i.e., $(\mathbf{E}, \mathbf{H})$. For $\alpha = 1$, above expressions will yield $\mathbf{E}_{fd} = \eta\mathbf{H}$ and $\eta\mathbf{H}_{fd} = -\mathbf{E}$, which are the dual field solutions of the original field solutions to the Maxwell's equations, i.e., $(\eta\mathbf{H}, -\mathbf{E}/\eta)$. For α between zero and one, $\mathbf{E}_{fd}$ and $\mathbf{H}_{fd}$ can be regarded as fractional dual solutions between the original fields and the dual fields.

Engheta considered a linearly polarized uniform TEM plane wave propagating along the positive z-direction [7]. The transverse fields associated with the wave are the original fields and are given as

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \exp(ikz)$$
$$\eta \mathbf{H} = \hat{\mathbf{y}} E_0 \exp(ikz).$$

Fractional dual fields for original fields given in the above set of field expressions are

$$\mathbf{E}_{fd} = \left[\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}} + \sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}}\right] E_0 \exp(ikz)$$
$$\eta \mathbf{H}_{fd} = \left[-\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}} + \cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}}\right] E_0 \exp(ikz).$$

It is obvious that fractional dual fields represent another uniform plane wave propagating in positive z-direction. However, its transverse fields are get rotated in xy-plane by an angle $(\alpha\pi/2)$ in the counterclockwise direction. In this discussion, it is planed to determine the fractional dual solutions for a TEM plane wave propagating in an arbitrary direction. Field radiated from an electric line source is termed as Green's function. It may be noted that Green's function may be represented as a spectrum of TEM plane waves propagating in different directions. So fractional dual solutions for TEM plane wave are utilized to calculate the fractional dual solutions corresponding to the field radiated from a line source.

It is well known that field radiated by a plate source is termed as one dimensional Green's function while field radiated by a line source is termed as two dimensional Green's function. Throughout the discussion electric fields radiated by the plate source and line source are represented as $\mathbf{G}_1$ and $\mathbf{G}_2$ respectively. Engheta [5] has shown that electric field radiated from a line source and field radiated from a plate source, in the far-zone along the axis of symmetry, can be related via a fractional order integral operator. Using fractional order integral relation, he derived solutions to the Helmholtz's equation that can be regarded as intermediate step between one dimensional and two dimensional Green's functions. This field may be termed as intermediate field. In the present discussion efforts are made to answer the following questions. Is the fractional dual field corresponding to a plate source and a line source, i.e., $\mathbf{G}_{1fd}$ and $\mathbf{G}_{2fd}$ respectively, can also be related via fractional order integral operator? What is the intermediate field between the two fractional dual fields $\mathbf{G}_{1fd}$ and $\mathbf{G}_{2fd}$? This field is termed as intermediate fractional dual field. Efforts are also made to find the source distributions corresponding to fractional dual solutions and intermediate fractional dual solutions.

2. TEM PLANE WAVE PROPAGATING IN AN ARBITRARY DIRECTION

Consider a unit amplitude TEM plane wave propagating in direction $\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$. Electric field of the wave is directed along y-axis of the coordinate system. Propagation constant of the medium is $k = \omega\sqrt{\epsilon\mu} = \sqrt{k_x^2 + k_z^2}$. It is assumed that medium is lossless, homogeneous and isotropic. Electric and magnetic fields associated with the plane wave are given below

$$\mathbf{E} = \exp(ik_x x + ik_z z)\hat{\mathbf{y}} \tag{1a}$$

$$\mathbf{H} = \frac{1}{\omega\mu}\exp(ik_x x + ik_z z)\left[-k_z\hat{\mathbf{x}} + k_x\hat{\mathbf{z}}\right]. \tag{1b}$$

It is desired to calculate the fractional dual solutions ($\mathbf{E}_{fd}$, $\mathbf{H}_{fd}$) for TEM plane wave. In order to find the fractional dual solutions for the TEM plane wave it is appropriate first to determine the eigenvalues and eigenvectors of operator $(\mathbf{k}\times = \{k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}\}\times)$ in the space C^3. Eigenvalues and eigenvectors of the operator are given below

$$\begin{aligned}
\mathbf{A}_1 &= \frac{1}{\sqrt{2}}\left[\frac{ik_z}{k}\hat{\mathbf{x}} + \hat{\mathbf{y}} - \frac{ik_x}{k}\hat{\mathbf{z}}\right] \qquad && a_1 = ik \\
\mathbf{A}_2 &= \frac{1}{\sqrt{2}}\left[\frac{-ik_z}{k}\hat{\mathbf{x}} + \hat{\mathbf{y}} + \frac{ik_x}{k}\hat{\mathbf{z}}\right] \qquad && a_2 = -ik \\
\mathbf{A}_3 &= \frac{ik_x}{k}\hat{\mathbf{x}} + \frac{ik_z}{k}\hat{\mathbf{z}} && a_3 = 0.
\end{aligned}$$

Above three eigenvectors are orthonormal basis vectors in C^3 space. Thus an arbitrary given vector $\widetilde{\mathbf{E}}(k_y, k_z) = \widetilde{E}_x\hat{\mathbf{x}} + \widetilde{E}_y\hat{\mathbf{y}} + \widetilde{E}_z\hat{\mathbf{z}}$ of space C^3 may be written as linear combination of eigenvectors ($\mathbf{A}_i$, $i = 1, 2, 3,$) as

$$\begin{aligned}
\widetilde{\mathbf{E}} &= \widetilde{E}_x\hat{\mathbf{x}} + \widetilde{E}_y\hat{\mathbf{y}} + \widetilde{E}_z\hat{\mathbf{z}} \\
&= P\mathbf{A}_1 + Q\mathbf{A}_2 + R\mathbf{A}_3
\end{aligned}$$

where

$$P = \frac{1}{\sqrt{2}}\left[\widetilde{E}_y - \frac{i}{k}\left\{\widetilde{E}_x k_z - \widetilde{E}_z k_x\right\}\right]$$
$$Q = \frac{1}{\sqrt{2}}\left[\widetilde{E}_y + \frac{i}{k}\left\{\widetilde{E}_x k_z - \widetilde{E}_z k_x\right\}\right]$$
$$R = \frac{-i}{k}\left[\widetilde{E}_x k_x + \widetilde{E}_z k_z\right].$$

Operation of the operator $(\mathbf{k}\times)$ on $\widetilde{\mathbf{E}}$ yields following result

$$\mathbf{k}\times\widetilde{\mathbf{E}} = (a_1)P\mathbf{A}_1 + (a_2)Q\mathbf{A}_2 + (a_3)R\mathbf{A}_3.$$

Therefore one can write corresponding expression for fractional cross product as

$$(\mathbf{k}\times)^\alpha\widetilde{E} = (a_1)^\alpha P\mathbf{A}_1 + (a_2)^\alpha Q\mathbf{A}_2 + (a_3)^\alpha R\mathbf{A}_3. \quad (2)$$

It is obvious that $(a_1)^\alpha$, $(a_2)^\alpha$ and $(a_3)^\alpha$ are the eigenvalues of the fractional cross product operator $(\mathbf{k}\times)^\alpha$.

Operation of curl operator on (1a) yields the results

$$\text{curl}[\mathbf{E}] = \text{curl}[\hat{\mathbf{y}}\exp(ik_x x + ik_z z)] = (i\mathbf{k}\times)\mathbf{E}. \quad (3)$$

Therefore the fractional curl

$$\text{curl}^\alpha[\mathbf{E}] = \text{curl}^\alpha[\hat{\mathbf{y}}\exp(ik_x x + ik_z z)] = (i\mathbf{k}\times)^\alpha\mathbf{E}. \quad (4a)$$

It is obvious that fractionalization of the curl operation in (4a) means fractionalization of cross product operator $(i\mathbf{k}\times)$. From expressions (2) and (4a), one can write the following expression

$$\text{curl}^\alpha[\mathbf{E}] = (i\mathbf{k}\times)^\alpha\mathbf{E} = (ia_1)^\alpha P\mathbf{A}_1 + (ia_2)^\alpha Q\mathbf{A}_2 + (ia_3)^\alpha R\mathbf{A}_3. \quad (4b)$$

Now consider another TEM plane wave which is propagating in direction $\mathbf{k} = k_x\hat{\mathbf{x}} - k_z\hat{\mathbf{z}}$, i.e.,

$$\mathbf{E} = \exp(ik_x x - ik_z z)\hat{\mathbf{y}} \quad (5a)$$
$$\mathbf{H} = \frac{1}{\omega\mu}\exp(ik_x x - ik_z z)\left[k_z\hat{\mathbf{x}} + k_x\hat{\mathbf{z}}\right]. \quad (5b)$$

Relation corresponding to the present case, i.e., for expressions (5a) and (5b), can be obtained from the relations obtained for the previous case, i.e., for expressions (1a) and (1b), by replacing k_z with $-k_z$.

Fractional dual fields that can be regarded as an intermediate solutions between the original fields and the dual fields can be obtained using the following relation

$$\mathbf{E}_{fd} = \frac{1}{(ik)^\alpha}\text{curl}^\alpha\mathbf{E} = \frac{1}{(ik)^\alpha}(i\mathbf{k}\times)^\alpha\mathbf{E} \tag{6a}$$

$$\eta\mathbf{H}_{fd} = \frac{1}{(ik)^\alpha}\text{curl}^\alpha(\eta\mathbf{H}) = \frac{1}{(ik)^\alpha}(i\mathbf{k}\times)^\alpha(\eta\mathbf{H}). \tag{6b}$$

Fractional fields corresponding to the original electric fields given by (1a) and (5a) are

$$\mathbf{E}^+_{fd} = \exp(ik_xx+ik_zz)\left[\frac{-k_z}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}} + \cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} + \frac{k_x}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right] \tag{7a}$$

$$\mathbf{E}^-_{fd} = \exp(ik_xx-ik_zz)\exp(-i\pi\alpha)\left[\frac{-k_z}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}} + \cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} - \frac{k_x}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right]. \tag{7b}$$

It may be noted that expression with superscript sign + corresponds to case when wave is propagating in direction $\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$ while expression with superscript sign $-$ corresponds to case when wave is propagating in direction $\mathbf{k} = k_x\hat{\mathbf{x}} - k_z\hat{\mathbf{z}}$. It may be noted that for $\alpha = 0$ above set of expressions yield result $\mathbf{E}$ and for $\alpha = 1$ yield $\eta\mathbf{H}$. Fractional fields corresponding to the original magnetic fields given by (1b) and (5b) are

$$\eta\mathbf{H}^+_{fd} = \frac{k\exp(ik_xx+ik_zz)}{\omega\mu}\left[-\frac{k_z}{k}\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}} - \sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} + \frac{k_x}{k}\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right] \tag{8a}$$

$$\eta\mathbf{H}^-_{fd} = \frac{k\exp(-i\alpha\pi)\exp(ik_xx-ik_zz)}{\omega\mu}\left[\frac{k_z}{k}\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}} + \sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} + \frac{k_x}{k}\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right]. \tag{8b}$$

For $\alpha = 0$, expression (8) will yield the results $\mathbf{H}$ while for $\alpha = 1$ expression (8) will give the result $-\mathbf{E}$. For α between zero and one fields given in (7) and (8) can be regarded as an fractional dual fields between the original and dual fields of a TEM plane wave propagating in an arbitrary direction. It is obvious from (7) and (8) that fractional dual fields also represents a plane wave propagating in the same direction as the original wave. However its transverse fields are get rotated by an angle $(\alpha\pi/2)$.

2.1 Standing waves

Consider a TEM plane wave is incident on a perfectly conducting sheet of infinite extent. Sheet is located at $z = 0$. Transverse fields associated with the wave are given in (1). Total field for region $z < 0$ can be written as sum of incident and reflected field as

$$\begin{aligned}\mathbf{E}^{\text{tot}} &= \mathbf{E}^{\text{inc}} + \mathbf{E}^{\text{ref}} \\ \mathbf{H}^{\text{tot}} &= \mathbf{H}^{\text{inc}} + \mathbf{H}^{\text{ref}}.\end{aligned}$$

Reflected electric and magnetic fields are negative of (5a) and (5b). Standing waves due to the original fields are

$$\begin{aligned}\mathbf{E}^{\text{tot}} &= \hat{\mathbf{y}} 2i \exp(ik_x x) \sin(k_z z) \\ \mathbf{H}^{\text{tot}} &= \frac{2\exp(ik_x x)}{\omega\mu} \left\{-k_z \hat{\mathbf{x}} \cos(k_z z) + ik_x \hat{\mathbf{z}} \sin(k_z z)\right\}.\end{aligned}$$

From the above expressions, surface impedance is given as

$$Z_{xy} = \frac{-E_y}{H_x} = \frac{i\omega\mu}{k_z} \tan(k_z z) = 0.$$

Fractional dual solutions corresponding to the above set of original solutions, i.e., $(\mathbf{E}^{\text{tot}}, \mathbf{H}^{\text{tot}})$, is given below

$$\begin{aligned}\mathbf{E}_{fd}^{\text{tot}} &= \mathbf{E}_{fd}^{+} - \mathbf{E}_{fd}^{-} \\ &= 2\exp(ik_x x) \exp\left(\frac{-i\alpha\pi}{2}\right) \left[\frac{-ik_z}{k} \sin\left(k_z z + \frac{\alpha\pi}{2}\right) \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{x}} \right. \\ &\quad \left. + i\sin\left(k_z z + \frac{\alpha\pi}{2}\right) \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} + \frac{k_x}{k} \cos\left(k_z z + \frac{\alpha\pi}{2}\right) \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{z}}\right]\end{aligned}$$

$$\begin{aligned}\mathbf{H}_{fd}^{\mathrm{tot}} &= \mathbf{H}_{fd}^{+} - \mathbf{E}_{fd}^{-} \\ &= \frac{2k\exp(ik_x x)}{\omega\mu}\exp\left(\frac{-i\alpha\pi}{2}\right)\left[\frac{-k_z}{k}\cos\left(k_z z + \frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}}\right. \\ &\quad \left. -\cos\left(k_z z + \frac{\alpha\pi}{2}\right)\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} + \frac{ik_x}{k}\sin\left(k_z z + \frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right].\end{aligned}$$

Surface impedance derived from above expressions is

$$Z_{yx} = \frac{-E_y}{H_x} = \frac{i\omega\mu}{k_z}\tan\left(\frac{\alpha\pi}{2}\right).$$

For $\alpha = 0$, $Z_{yx} = 0$ which indicates that boundary is a PEC and for $\alpha = 1$, $Z_{yx} = \infty$ so boundary is PMC (perfect magnetic conductor). For α between zero and one, surface impedance lies between PEC and PMC.

3. SPECTRUM OF TEM PLANE WAVES

Consider an electric line source of infinite extent in y-direction. Line source is located at origin of the coordinate system. Electric field radiated from the electric line source can be represented as spectrum of plane waves and is given below

$$\mathbf{G}_2(x, z; 0, 0) = \frac{-\omega\mu}{4\pi}\hat{\mathbf{y}}\int_{-\infty}^{\infty}\frac{1}{k_x}\exp(ik_x x + ik_z z)dk_z, \quad x > 0. \tag{9}$$

It may be noted that above expression can be regarded as a spectrum of TEM plane waves propagating in different directions. Application of curl operation on the above expression yields following result

$$\mathrm{curl}[\mathbf{G}_2(x, z; 0, 0)] = \frac{-\omega\mu}{4\pi}\int_{-\infty}^{\infty}\frac{1}{k_x}[(i\mathbf{k}\times)\hat{\mathbf{y}}]\exp(ik_x x + ik_z z)dk_z$$

where

$$\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}.$$

Fractionalization of the curl operator means the fractionalization of the cross product operator, $(i\mathbf{k}\times)$, i.e.,

$$\mathrm{curl}^{\alpha}[\mathbf{G}_2(x, z; 0, 0)] = \frac{-\omega\mu}{4\pi}\int_{-\infty}^{\infty}\frac{1}{k_x}[(i\mathbf{k}\times)^{\alpha}\hat{\mathbf{y}}]\exp(ik_x x + ik_z z)dk_z.$$

Substituting the results obtained from the above expression into following expression

$$\mathbf{G}_{2fd} = \frac{1}{(ik)^\alpha} \mathrm{curl}^\alpha \mathbf{G}_2 \tag{10}$$

yields the following

$$\mathbf{G}_{2fd} = \frac{-\omega\mu}{4\pi} \int_{-\infty}^{\infty} \frac{1}{k_x} \exp(ik_x x + ik_z z) \left[-\frac{k_z}{k} \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{x}} + \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} + \frac{k_x}{k} \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{z}} \right] dk_z.$$

Above expression provides one out of the two components in the set for fractional dual solutions. This component is obtained from the original electric field $\mathbf{G}_2$. Solving the above expression asymptotically [8] for $k\rho \to \infty$ and retaining only dominant term following result is obtained

$$\mathbf{G}_{2fd} \sim \frac{-\omega\mu}{\sqrt{2\pi}} \frac{\exp(ik\rho - i\pi/4)}{\sqrt{k\rho}} \left[-\cos\theta \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{x}} + \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} + \sin\theta \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{z}} \right]. \tag{11}$$

Similarly other component in fractional dual solution set can be derived.

4. SOURCE DISTRIBUTION

Consider an infinite plate source located at $\delta(x)$. The current distribution on plate is $\mathbf{J}_1 \exp(-i\omega t) = -i\omega\mu\delta(x) \exp(-i\omega t)\hat{\mathbf{y}}$. For this current distribution, plate source will radiate plane waves propagating in the x-direction. Electric field radiated by the plate source is

$$\mathbf{G}_1 = \frac{-\omega\mu}{2k} \exp(ikx)\hat{\mathbf{y}}, \qquad x > 0.$$

Fractional dual field corresponding to the above original electric field is

$$\mathbf{G}_{1fd} = \frac{-\omega\mu}{2k} \exp(ikx) \left\{ \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} + \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{z}} \right\}. \tag{12}$$

An important and interesting question occurs is that what would be the current distribution corresponding to the fractional dual fields of a plate source? In order to calculate the corresponding current distribution, consider the following Helmholtz's equation

$$\nabla^2 \mathbf{G_1} + k^2 \mathbf{G_1} = -i\omega\mu\delta(x)\hat{\mathbf{y}}. \tag{13}$$

Taking the curl of both sides yields the following

$$\nabla^2 \text{curl}\mathbf{G}_1 + k^2 \text{curl}\mathbf{G}_1 = -i\omega\mu[\text{curl}\delta(x)\hat{\mathbf{y}}].$$

From above expression one can write

$$\nabla^2 \mathbf{G}_{1fd} + k^2 \mathbf{G}_{1fd} = -\frac{i\omega\mu}{(ik)^\alpha}\text{curl}^\alpha \delta(x)\hat{\mathbf{y}} = \mathbf{J}_{1fd}.$$

Dirac delta function may be written in terms of Fourier spectrum as

$$\mathbf{J}_1 = -i\omega\mu\delta(x)\hat{\mathbf{y}} = \frac{-i\omega\mu}{2\pi}\hat{\mathbf{y}} \int_{-\infty}^{\infty} \exp(ik_x x)dk_z.$$

Taking the fractional curl of both sides yields the following

$$\text{curl}^\alpha \mathbf{J}_1 = -i\omega\mu\text{curl}^\alpha \delta(x)\hat{\mathbf{y}} = \frac{-i\omega\mu}{2\pi} \int_{-\infty}^{\infty} [(i\mathbf{k}\times)^\alpha \hat{\mathbf{y}}] \exp(ik_x x)dk_z.$$

Therefore using above results source distribution $\mathbf{J}_{1fd}$ corresponding to fractional dual fields of a plate source can be obtained as

$$\mathbf{J}_{1fd} = \frac{-i\omega\mu}{2\pi} \int_{-\infty}^{\infty} \left[\exp(ik_x x) \left\{\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} + \sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right\}\right] dk_z.$$

Taking the inverse Fourier transform of above expression

$$\mathbf{J}_{1fd} = -i\omega\mu\delta(x)\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} - i\omega\mu\delta(x)\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}. \tag{14}$$

First term in the above expression may be considered as positive y-directed uniform current distribution on an infinite plate. Second term in the above expression may be also considered as positive z-directed current distribution on an infinite plate. It is obvious from (14) that fractional sources $\mathbf{J}_{1fd}$ can be obtained from $\mathbf{J}_1$ by rotating the direction of current through an angle of $(\alpha\pi/2)$ in counterclockwise direction.

Now calculate the source distribution $\mathbf{J}_{2fd}$ corresponding to the fractional dual fields of a line source. Source distribution $\mathbf{J}_{2fd}$ is obtained as

$$\mathbf{J}_{2fd} = \frac{-i\omega\mu}{4\pi} \int_{-\infty}^{\infty} \exp(ik_x x + ik_z z) \left[-\frac{k_z}{k} \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{x}} + \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} + \frac{k_x}{k} \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{z}} \right] dk_z.$$

Above expression may be written as

$$\mathbf{J}_{2fd} = \frac{-i\omega\mu}{4\pi k} \sin(\frac{\alpha\pi}{2}) \int_{-\infty}^{\infty} [\mathbf{k} \times \hat{\mathbf{y}}] \exp(ik_x x + ik_z z) dk_z - i\omega\mu\delta(x)\delta(z) \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} \tag{15}$$

$$\mathbf{J}_{2fd} = \frac{-\omega\mu}{k} \sin\left(\frac{\alpha\pi}{2}\right) \mathrm{curl}\, [\delta(x)\delta(z)\hat{\mathbf{y}}] - i\omega\mu\delta(x)\delta(z) \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}}. \tag{16}$$

Source distributions $\mathbf{J}_{1fd}$ and $\mathbf{J}_{2fd}$ may be termed as fractional dual source distributions for plate source and line source respectively.

Engheta [5] using the tools of fractional calculus proposed some solutions to the scalar Helmholtz's equation that can be regarded as an intermediate step between the fields radiated from a plate source and radiated fields from a line source. Corresponding source distribution that can be regarded as an intermediate step between the source distribution on a plate source $\mathbf{J_1}$ and a line source $\mathbf{J_2}$ is given as [5]

$$ {}_eJ_{f_1}(x, z) = \frac{-i\omega\mu\delta(x)|z|^{1-f_1}}{2\Gamma(2 - f_1)} \quad \text{for} \quad 1 < f_1 < 2.$$

Source distribution in the above expression is termed as intermediate source distribution. It may be noted that above expression for the limiting case of $f_1 = 1$ yields $J_1/2$ while yields J_2 for the limiting case of $f_1 = 2$. It may be noted that source distribution given in the above expression is even symmetric about the xy-plane. Source distribution that can be regarded as intermediate step between the current distributions corresponding to fractional dual fields of a plate source and fractional dual fields of a line source is obtained as

$$\mathbf{J}_{f_1 fd} = \frac{1}{k} \sin\left(\frac{\alpha\pi}{2}\right) \mathrm{curl}\, [{}_eJ_{f_1}\hat{\mathbf{y}}] + \cos\left(\frac{\alpha\pi}{2}\right) [{}_eJ_{f_1}\hat{\mathbf{y}}] \qquad 1 < f_1 < 2 \quad \text{and} \quad 0 < \alpha < 1. \tag{17}$$

It is obvious that for limit of $f_1 = 2$, ${}_eJ_{f_1}$ reduces to J_2 and ${}_e\mathbf{J}_{f_1fd}$ reduces to $\mathbf{J}_{2fd}$. The limit of $f_1 = 1$, ${}_eJ_{f_1}$ reduces to $J_1/2$ and ${}_e\mathbf{J}_{f_1fd}$ reduces to $\mathbf{J}_{1fd}/2$. Source distribution in the expression (17) is termed as intermediate fractional dual source distribution.

5. INTERMEDIATE FRACTIONAL DUAL SOLUTIONS TO THE HELMHOTZ'S EQUATION

It has been shown in [5, 9] that electric field radiated from a line source $\mathbf{G}_2$ and a plate source $\mathbf{G}_1$, in the far-zone along the axis of symmetry, are related via fractional order integral operator. The relation between field radiated $\mathbf{G}_2$ and $\mathbf{G}_1$ in a homogeneous space having propagation constant k is given as [5]

$$\mathbf{G}_2(x, z = 0) \approx \frac{1}{2\sqrt{\pi}} {}_0D_{k^2}{}^{-1/2}\mathbf{G}_1$$

where the order of the integral operator is $(-1/2)$. Variable of integration is k^2 and 0 is lower limit.

In this section, it is desired to note that whether the fractional dual field corresponding to line source, i.e., $\mathbf{G}_{2fd}$, and corresponding to plate source, i.e., $\mathbf{G}_{1fd}$, are also related via a fractional order integral operator? Fractional dual radiated field from the line source along the axis of symmetry $(z = 0)$ and in the far-zone is

$$\mathbf{G}_{2fd}(x, z = 0) \approx \frac{-\omega\mu}{2\pi} \int_0^k \frac{1}{k_x} \exp(ik_x x) \left[\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} + \frac{k_x}{k} \sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right] dk_z.$$

Using change of variable $\sqrt{k^2 - k_z^2} = u$ in the above expression yields the following

$$\mathbf{G}_{2fd} \approx \frac{-\omega\mu}{2\pi} \int_0^k \frac{1}{\sqrt{k^2 - u^2}} \exp(iux) \left[\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} + \frac{u}{k} \sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right] du.$$

One more change of variable $u^2 = w$ in the above equation yields the following

$$\mathbf{G}_{2fd} \approx \frac{-\omega\mu}{2\pi} \int_0^{k^2} \frac{1}{\sqrt{k^2 - w}} \frac{\exp(i\sqrt{w}x)}{2\sqrt{w}} \left[\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}} + \frac{\sqrt{w}}{k} \sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right] dw.$$

It is obvious from above expression that fractionalized electric field radiated from a line source, in the far-zone along the axis of symmetry, can be expressed in terms of fractional order integral of a quantity as

$$\mathbf{G}_{2fd}(x, z=0) \approx \frac{1}{2\sqrt{\pi}} {}_0D_{k^2}{}^{-1/2} \left[\frac{-\omega\mu}{2k} \exp(ikx) \left\{ \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} + \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{z}} \right\} \right].$$

Quantity

$$\mathbf{G}_{1fd} = \frac{-\omega\mu}{2k} \exp(ikx) \left\{ \cos\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} + \sin\left(\frac{\alpha\pi}{2}\right) \hat{\mathbf{z}} \right\}$$

is fractional dual field corresponding to the plate source located at $\delta(x)$. This means that above expression may be written as

$$\frac{1}{(ik)^\alpha} \text{curl}^\alpha \mathbf{G}_2(x, z=0) \approx \frac{1}{2\sqrt{\pi}} {}_0D_{k^2}{}^{-1/2} \left[\frac{1}{(ik)^\alpha} \text{curl}^\alpha \mathbf{G}_1 \right]$$
$$\mathbf{G}_{2fd}(x, z=0) \approx \frac{1}{2\sqrt{\pi}} {}_0D_{k^2}{}^{-1/2} \mathbf{G}_{1fd}.$$

It is concluded from the above expression that fractional dual field of line source and a plate source, in the far-zone along the axis of symmetry, are also related via fractional order integral operator.

In the remaining part of this section, efforts are made to find the field that can be regarded as intermediate step between the fractional dual field of a plate source and fractional dual field of a line source. Variable $f_1 = (1-2\beta)$ is introduced such that, when order β varies between 0 and $-1/2$ the variable f_1 sweeps a range between one and two. Introduction of variable f_1 modifies above expression to the following form

$$\mathbf{G}_{f_1 fd}(x, z=0) \approx \frac{1}{2\sqrt{\pi}} {}_0D_{k^2}{}^{(1-f_1)/2} \mathbf{G}_{1fd}.$$

It is important to note that although above expression yields result that is for the axis of symmetry with points of observations in the far-zone but these results can be very easily converted to a form which is valid for the whole region. Expression will be approximately the same as for $\mathbf{G}_{2fd}$ except that one pole will get introduce at $k_z = 0$ as

$$\mathbf{G}_{f_1fd} = \frac{-\omega\mu}{4\pi}\int_{-\infty}^{\infty}\frac{1}{k_x}\frac{1}{(ik_z)^{2-f_1}}\exp(ik_xx+ik_zz)\left[-\frac{k_z}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}}\right.$$
$$\left.+\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}}+\frac{k_x}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right]dk_z,$$
$$0<\alpha<1, \qquad 1<f_1<2.$$

For the symmetric source distribution ${}_eJ_{f_1fd}$, which can be considered as an intermediate step between J_{1fd} and J_{2fd}, solution to the Helmholtz's equation can be written as

$$\begin{aligned}{}_e\mathbf{G}_{f_1fd} &= \frac{-\omega\mu}{8\pi}\int_{-\infty}^{\infty}\frac{1}{k_x}\frac{1}{(ik_z)^{2-f_1}}\exp(ik_xx+ik_zz)\\ &\quad\left[-\frac{k_z}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}}+\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}}+\frac{k_x}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right]dk_z\\ &\quad+\frac{-\omega\mu}{8\pi}\int_{-\infty}^{\infty}\frac{1}{k_x}\frac{1}{(ik_z)^{2-f_1}}\exp(ik_xx-ik_zz)\\ &\quad\left[-\frac{k_z}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}}+\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}}+\frac{k_x}{k}\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right]dk_z.\end{aligned}$$

Using the change of variables, from Cartesian coordinate system to cylindrical coordinate system, by the following transformations

$$z=\rho\cos\theta \qquad k_z=k\cos\phi$$
$$y=\rho\sin\theta \qquad k_x=k\sin\phi$$

above expression reduces to the following form

$$I_{\frac{1}{2}} = \frac{-\omega\mu}{8\pi}\int_C \mathbf{A}(\phi)\exp\{\pm ik\rho\cos(\theta\mp\phi)\}d\phi \tag{18}$$

where

$$\mathbf{A}(\phi)=\frac{1}{(ik\cos\phi)^{2-f_1}}\left[-\sin(\phi)\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{x}}\right.$$
$$\left.+\cos\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{y}}+\cos(\phi)\sin\left(\frac{\alpha\pi}{2}\right)\hat{\mathbf{z}}\right]$$

where C is the contour in the complex ϕ-plane. Integrals in (18) can be calculated using asymptotic technique [8]. First consider the integral I_1. If the point of observation lies in range $x > 0$, i.e., $0 < \theta < \pi/2$, the deformed path, i.e., steepest decent path, will not intersect the branch cut at $\phi = \pi/2$. When the observation point lies in range $\pi/2 < \theta < \pi$ the deformed path will intersect the branch cut at $\phi = \pi/2$. Therefore the contribution along the branch cut around the branch point is required to note the additional contributions due to the intermediate source [5]. The asymptotic contribution due to integral I_2 can calculated by replacing θ with $\theta + \pi$ and z with $-z$ in the asymptotic expression for I_1. The far-zone radiated fields when the observation point is not too close to $\theta = \pi/2$ is given by the following expression The far-zone radiated fields when the observation point is not too close to $\theta = \pi/2$ is given by the following expression

$$
{}_e\mathbf{G}_{f_1fd}(x,z) \sim -\cos(\pi f_1/2)(k|\cos\theta|)^{f_1-2}\mathbf{G}_{2fd} + \frac{1}{2\Gamma(2-f_1)k^{1-f_1}}\frac{\mathbf{G}_{1fd}}{(k|z|)^{f_1-1}}.
$$

It may be noted that in the above expression $\mathbf{G}_{1fd}$ and $\mathbf{G}_{2fd}$ represents far-zone fractional dual fields of one and two dimensional Green's functions $\mathbf{G}_1$ and $\mathbf{G}_2$ respectively. Far-zone field expression for $\mathbf{G}_{2fd}$ and $\mathbf{G}_{1fd}$ are (11) and (12) respectively.

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