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PULSE PROPAGATION IN SEA WATER: THE MODULATED PULSE

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1. INTRODUCTION

Electromagnetic fields have long been recognized as a useful tool for geophysical exploration and remote sensing [1, 2]. Over sixty years ago, geophysical techniques, particularly the Eltran methods, involved the excitation of antennas by current pulses and the subsequent measurement of the transient scattered fields in order to extract qualitative information about the dielectric properties of different layers inside the earth [3–5]. In more recent years, electromagnetic pulses are studied in connection with the optimization of the detection and discrimination of scatterers submerged in dissipative media—specifically, conducting earth and sea water [6, 7]—and even the diagnosis and treatment of pathologies in biological tissues [8].

When a pulse travels in a dissipative medium, its shape along with its characteristics (amplitude, duration, rise and decay time) are modified because the complex wave number is no longer linear in frequency. Specifically, the large value of conductivity σ in sea water renders desirable for remote sensing, as well as for communication with submarines, the application of signals whose spectrum has significant components only at very low frequencies. Within this frequency range, a transmitting antenna positioned vertically to and over the boundary between air and sea becomes physically impractical because it may not be made as long as needed to maintain useful properties in the frequency domain, an example of such properties being the sufficiently high value of the radiation resistance [9].

A configuration that is often proposed consists of a transmitting horizontal antenna just below the sea surface [9], where the receiver is chosen to be not too close to the transmitter. Accordingly, the incoming signal results from the interference of a pulse that reaches the receiver after propagating downward into the sea, being reflected, and then propagating upward, with a pulse that travels along the interface in a surface wave. Of course, the useful signal should be optimally detected via continuous cancellation of the second pulse. This task should require some rather specialized signal-processing techniques and lies beyond the scope of the present paper.

A question that needs to be addressed first is: What is the form of the electromagnetic pulse that travels downward into the sea at any practical distance from the source, when the current excitation is a realistic function of time? A quantitative answer to this question is a step toward an understanding of how realizable pulses propagate and scatter in highly conducting media and may offer some insight into certain physical limitations associated with remote sensing. In order to simplify the analysis in this paper, the horizontal antenna is replaced by an infinitesimal electric dipole oriented along the z-axis and immersed in an infinite medium in the absence of the receiver. The geometry is shown in Fig. 1.

Early advanced analyses of pulse propagation in dispersive media by Sommerfeld [10] and Brillouin [11] were motivated by the question of defining and calculating a signal velocity. These authors studied, asymptotically in distance, the propagation of one-dimensional plane-wave pulses caused by the sinusoidal modulation of time step-discontinuous boundary conditions in a medium characterized by the Lorentz dispersion formula [12]. (See also [13, 14].) The problem and

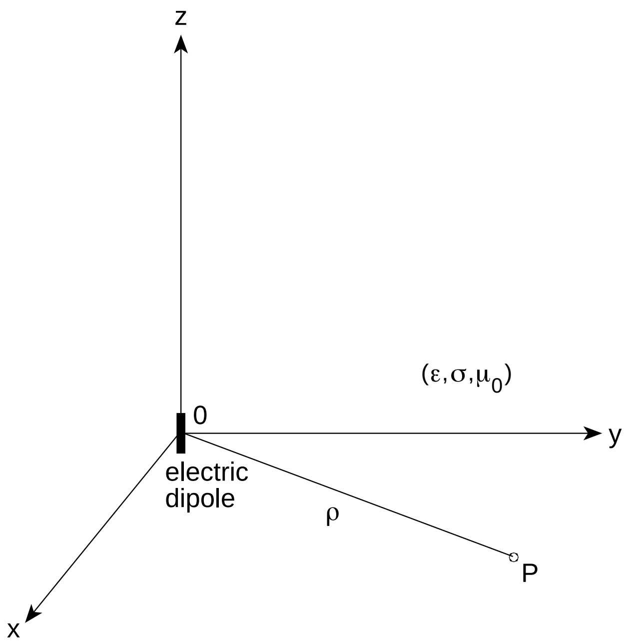

Figure 1. The geometry of the problem. The dipole is immersed in an infinite medium of dielectric permittivity ϵ, conductivity σ, and magnetic permeability μ_0. P is the observation point in the equatorial plane of the dipole.

methodology of Sommerfeld and Brillouin were later revisited by others [15–18].

Noteworthy is the much discussed recent work by Oughstun and co-workers, who applied so-called "uniform asymptotic expansions" without any apparent restriction to low frequencies [19–21]. In these analyses attention again focused on plane-wave pulses for sufficiently large values of the propagation distance.

The transient fields of electric dipoles, however, may not be approximated by plane waves produced by infinite current sheets. Additional complications arise in the frequency domain because a dipole source in an infinite space creates a field of interest that has a complicated structure, shifting progressively from the near to the far fields.

The propagation of pulses excited by Hertzian dipoles was examined by Wait [1, 22] and Wait and Spies [23, 24], and was later studied systematically under a low-frequency approximation by King [7, 25] and King and Wu [26] in relation to remote sensing in sea water. Their approximations are based on the condition $\sigma \gg \omega\epsilon$, valid for all frequencies of interest in sea water. More precisely, with $\epsilon \simeq 80\epsilon_0$, which is the static value of the dielectric permittivity at 20° C [27], and $\sigma \simeq 4$ S/m, the ensuing condition on the frequency f is

$$f \ll \sigma/(2\pi\epsilon) = 9.0 \times 10^8 \,\text{Hz}, \tag{1.1}$$

which poses no practical restriction. Notably, in these works the current pulse is either a modulated rectangular pulse with an envelope of zero rise and decay time [25, 26] or a pulse with a gaussian envelope extending from $-\infty$ to $+\infty$ [7]. The problem of the effect of a finite rise and decay time was addressed by Margetis soon after for the more realistic excitation pulse [28]

$$p(t) = \frac{1}{T_\text{w}}\{(1 - e^{-\omega_{rs}t})u(t) - [1 - e^{-\omega_{rs}(t-T_\text{w})}]u(t - T_\text{w})\}. \tag{1.2}$$

It is the purpose of this paper to extend the analysis of [28] to the case of a *modulated* current pulse of the exponential type. The exposition is organized as follows. Section 2 revisits the necessary mathematical framework and underlying physical principles, with some emphasis on various subtleties related to the concept of classical causality and the validity of the low-frequency (or diffusion) approximation. In Sec. 3, the electric field in the equatorial plane of the dipole is calculated under the low-frequency approximation; the dipole is excited by a continuously differentiable source pulse of nonzero rise and decay times that is modulated by a carrier frequency ω_0. In Sec. 4, simple asymptotic formulas are derived for observation points sufficiently close to the source and times of practical interest.

2. FORMULATION

2.1 Preliminaries

As is depicted in Fig. 1, the $\hat{\mathbf{z}}$-directed dipole is located at the origin in an infinite medium with static conductivity $\sigma_\text{st} = \sigma$, static dielectric permittivity $\epsilon_\text{st} = \epsilon = \epsilon_r\epsilon_0$, and magnetic permeability $\mu = \mu_0$.

Maxwell's equations read

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \tag{2.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t), \tag{2.2}$$

where

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}_s(\mathbf{r}, t) + \mathbf{J}_c(\mathbf{r}, t), \tag{2.3}$$

$$\mathbf{J}_s(\mathbf{r}, t) = 2h_e I_0 p(t)\delta(\mathbf{r})\hat{\mathbf{z}}, \qquad p(t < 0) = 0, \tag{2.4}$$

is the source current density, and $\mathbf{J}_c(\mathbf{r}, t)$ is the conduction current density. $p(t)$ is taken to be bounded with $p(t) \leq O(e^{-\varsigma t})$ for $t \to \infty$ $(\varsigma > 0)$. For the frequency range of interest, the constitutive relations are assumed to be

$$\mathbf{J}_c(\mathbf{r}, t) = \sigma \mathbf{E}(\mathbf{r}, t), \tag{2.5a}$$

$$\mathbf{D}(\mathbf{r}, t) = \epsilon \mathbf{E}(\mathbf{r}, t), \qquad \mathbf{H}(\mathbf{r}, t) = \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}, t). \tag{2.5b}$$

By virtue of the equation of continuity, $-\nabla \cdot \mathbf{J} = \partial\rho/\partial t$, it follows that

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}(\mathbf{r}, t)}{\partial t^2} = \frac{1}{\epsilon} \nabla \rho_s(\mathbf{r}, t) + \mu_0 \frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t}, \tag{2.6}$$

provided that at some fixed time $t = t_0$,

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t_0) = \rho(\mathbf{r}, t_0)/\epsilon. \tag{2.7}$$

In the above, $v = (\mu_0 \epsilon)^{-1/2}$ is the velocity of light in the medium under consideration and $\rho_s(\mathbf{r}, t)$ is the source charge density.

It is mathematically convenient to assume the Fourier representation

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_C d\omega \, e^{-i\omega t} \, \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega), \quad \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega : \text{real}) = \boldsymbol{\mathcal{E}}^*(\mathbf{r}, -\omega). \tag{2.8}$$

What can be said *a priori* about the contour C is that it should be asymptotically parallel to the real axis and *above* all singularities of $\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$ in accord with the fundamental ideas of classical causality.

This means that at any time t the field quantities for $r > 0$ are determined only by values of $\mathbf{J}_s(\mathbf{r}, t)$ prior to t [12]. In particular,

$$\text{if} \quad p(t) = \delta(t), \quad \text{then} \quad \mathbf{E}(\mathbf{r}, t) = 0 \quad \text{for} \quad t < 0, \tag{2.9}$$

$\mathbf{E}(\mathbf{r}, t)$ corresponding to the known retarded Green's function for a lossless medium when $\sigma = 0$. Attention now focuses on the $\mathbf{E}(\mathbf{r}, t)$ for which $\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$ is holomorphic and bounded in the upper half of the ω-plane. Contour C may be taken to coincide with the real axis except possibly in the neighborhood of singular points, such as real poles, where C has to be indented in the upper ω-plane. If the component $\mathcal{E}_i(\mathbf{r}, \omega)$ equals 1, $E_i(\mathbf{r}, t)$ becomes the δ-function $\delta(t)$ ($i = x, y, z$). Figure 2 shows the integration path for $p(t) = \delta(t)$.

Clearly, $\mathbf{E}(\mathbf{r}, t)$ satisfies Eq. (2.6) for $|\mathbf{r}| > 0$ if $\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$ obeys the homogeneous Helmholtz equation

$$\nabla^2 \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) + k(\omega)^2 \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) = 0, \quad r > 0, \tag{2.10}$$

where the complex wave number $k(\omega)$ is

$$k(\omega) = \sqrt{\omega^2 \mu_0 \epsilon + i\omega\mu_0\sigma} = e^{i\pi/4}\sqrt{\omega\mu_0\sigma}\,\sqrt{1 - i\omega\epsilon/\sigma} \tag{2.11}$$

and a Riemann sheet for the square root is chosen so that

$$\lim_{|\omega|\to\infty} \left[k(\omega) - \frac{\omega}{v}\right] = \frac{i}{2}\,\sigma\sqrt{\frac{\mu_0}{\epsilon}}, \tag{2.12}$$

with the branch-cut configuration of Fig. 2. Of course, contour C should lie entirely in this first Riemann sheet.

The calculation of fields is facilitated via the introduction in the time domain of the conventional vector potential $\mathbf{A}(\mathbf{r}, t)$. With the condition

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{1}{v^2}\frac{\partial \Phi(\mathbf{r}, t)}{\partial t} + \mu_0 \sigma \Phi(\mathbf{r}, t) = 0 \tag{2.13}$$

and

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \nabla \Phi(\mathbf{r}, t), \tag{2.14}$$

the Fourier transform $\boldsymbol{\mathcal{A}}(\mathbf{r}, \omega)$ of $\mathbf{A}(\mathbf{r}, t)$ is immediately recognized as the Green's function for the Helmholtz equation, with the well-known admissible solution [29]

$$\boldsymbol{\mathcal{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}}\, 2h_e I_0\, \frac{\mu_0}{4\pi}\, \wp(\omega)\, \frac{e^{ik(\omega)r}}{r}. \tag{2.15}$$

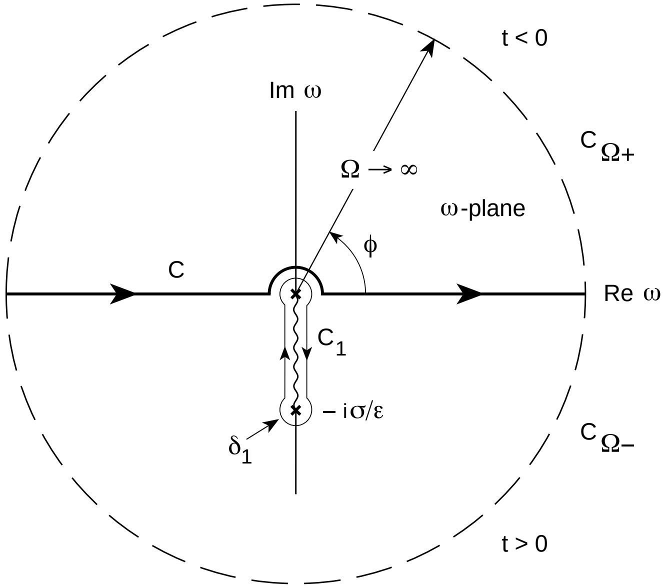

Figure 2. Branch-cut configuration in the ω-plane for $k(\omega)$ introduced in (2.10). Contour C is the original path of integration for the impulse response of the dipole and C_1 is the modified path for $t > 0$. $C_{\Omega\pm}$ are large semi-circles in the upper $(+)$ or lower $(-)$ half-plane. δ_1 is the radius of each small circle around the branch points at $\omega = 0, -i\sigma/\epsilon$.

For $\sigma = 0$, condition (2.13) amounts to choosing the usual Lorentz gauge. $\wp(\omega)$ is defined by

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \wp(\omega). \tag{2.16}$$

The electric field $\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$ in free space is readily obtained as

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) &= \frac{i\omega}{k(\omega)^2} \nabla \times \nabla \times \boldsymbol{\mathcal{A}}(\mathbf{r}, \omega) \\ &= -\frac{\omega\mu_0}{4\pi k(\omega)^2}\, 2h_e I_0\, \wp(\omega) e^{ik(\omega)r} \left\{ \hat{\boldsymbol{\rho}}\, \frac{\rho z}{r^2} \left[\frac{ik(\omega)^2}{r} - \frac{3k(\omega)}{r^2} - \frac{3i}{r^3} \right] \right. \\ &\quad \left. - \hat{\mathbf{z}} \left[\frac{2k(\omega)}{r^2} + \frac{2i}{r^3} + \frac{\rho^2}{r^2} \left(\frac{ik(\omega)^2}{r} - \frac{3k(\omega)}{r^2} - \frac{3i}{r^3} \right) \right] \right\}, \end{aligned} \tag{2.17}$$

where (ρ, ϕ, z) are cylindrical coordinates. In the equatorial plane of the dipole, $z = 0$, the complex electric field is of maximum magnitude:

$$\begin{aligned}\boldsymbol{\mathcal{E}}&(x, y, z = 0, \omega) \\ &= \hat{\mathbf{z}}\mathcal{E}_z(\rho, \omega) \\ &= \hat{\mathbf{z}}\,\frac{\omega\mu_0}{4\pi}\,(2h_e I_0)\wp(\omega)e^{ik(\omega)\rho}\left[\frac{i}{\rho} - \frac{1}{k(\omega)\rho^2} - \frac{i}{k(\omega)^2\rho^3}\right]. \quad (2.18)\end{aligned}$$

2.2 The Low-Frequency Approximation

Since only low frequencies are of interest in sea water, it is reasonable to invoke condition (1.1) and approximate $k(\omega)$ as

$$k(\omega) \sim (i\omega\mu_0\sigma)^{1/2} = (1+i)a\sqrt{\omega}, \qquad a = \sqrt{\mu_0\sigma/2}, \tag{2.19}$$

in a Lorentz frame where the source does not move.

This approximation amounts to the neglect of the displacement current $-i\omega\epsilon\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$ in comparison with the conduction current $\sigma\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$. In the time domain, the fields are subject to the corresponding condition

$$\left|\frac{\partial\mathbf{D}(\mathbf{r}, t)}{\partial t}\right| \ll \sigma|\mathbf{E}(\mathbf{r}, t)|. \tag{2.20}$$

The sequence of steps that previously furnished Eq. (2.6) now leads to

$$\nabla^2\mathbf{E}^{\mathrm{lf}}(\mathbf{r}, t) = \mu_0\sigma\,\frac{\partial\mathbf{E}^{\mathrm{lf}}}{\partial t}, \quad \mathbf{E} \sim \mathbf{E}^{\mathrm{lf}}, \qquad r > 0, \tag{2.21}$$

which reduces to Fick's second (or diffusion) equation for each Cartesian component of $\mathbf{E}^{\mathrm{lf}}$. Analogies between the propagation of low-frequency pulses in highly conducting media and the conduction of heat in thermal bodies are easily drawn from this equation.

Equation (2.18) under approximation (2.19) gives

$$\mathcal{E}_z^{\mathrm{lf}}(\rho, \omega) = \frac{\mu_0}{2\pi}\,h_e I_0\,\wp(\omega)\left[\frac{i\omega}{\rho} - \frac{(1-i)\sqrt{\omega}}{2a\rho^2} - \frac{1}{2a^2\rho^3}\right]e^{ia\rho\sqrt{\omega}}e^{-a\rho\sqrt{\omega}}. \tag{2.22}$$

Accordingly, the time-dependent field reads

$$\begin{aligned}E_z^{\mathrm{lf}}(\rho, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\mathcal{E}_z^{\mathrm{lf}}(\rho, \omega) \\ &= \frac{\mu_0 h_e I_0}{4\pi}\left[\frac{I_1(\rho, t)}{\rho} + \frac{I_2(\rho, t)}{a\rho^2} + \frac{I_3(\rho, t)}{a^2\rho^3}\right], \quad (2.23)\end{aligned}$$

where

$$I_1(\rho,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, i\omega\wp(\omega)e^{-(1-i)a\rho\sqrt{\omega}}, \tag{2.24}$$

$$I_2(\rho,t) = \frac{i-1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \sqrt{\omega}\,\wp(\omega)e^{-(1-i)a\rho\sqrt{\omega}}, \tag{2.25}$$

$$I_3(\rho,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \wp(\omega)e^{-(1-i)a\rho\sqrt{\omega}}. \tag{2.26}$$

The integration path here is indented in the upper ω-plane around singularities that lie in the real axis, in accord with classical causality. The first Riemann sheet is chosen so that $\sqrt{\omega} > 0$ for $\omega > 0$ and the branch cut is taken to coincide with the negative imaginary axis, for instance, by letting $\epsilon \to 0^+$ in Fig. 2. Note that

$$I_2(\rho,t) = -\frac{1}{a}\,\frac{\partial I_3(\rho,t)}{\partial\rho}, \tag{2.27a}$$

$$I_1(\rho,t) = 2\,\frac{\partial I_3(\rho,t)}{\partial t}, \tag{2.27b}$$

by interchanging the order of integration and differentiation on the grounds of an assumed uniform in t and $\rho > 0$ convergence of the integrals for $I_1(\rho,t)$, $I_2(\rho,t)$, and $I_3(\rho,t)$. Hence, it is sufficient to evaluate only $I_3(\rho,t)$.

A word of caution about Eq. (2.21) in comparison to the original Eq. (2.6) is now in order. While the latter allows only for electromagnetic pulses that propagate with velocity which is not greater than the (finite) velocity of light $(\mu_0\epsilon)^{-1/2}$, the former admits solutions that propagate instantaneously everywhere in space. However, the use of this equation should pose no practical limitation for distances and times of interest in sea water.

As is discussed in more technical detail in [30], approximation (2.19) in the frequency domain implies minimally the conditions

$$t \gg \rho/v, \quad t \gg \tau_0 = \epsilon/\sigma, \quad t \gg \frac{\rho}{v}(\zeta\rho)^{1/3} \tag{2.28}$$

in the time domain. Because

$$\tau_0 \simeq 0.18 \text{ ns}, \quad \zeta \equiv -i \lim_{|\omega|\to\infty}\left[k(\omega) - \frac{\omega}{v}\right] = \frac{1}{2}\sigma\sqrt{\frac{\mu_0}{\epsilon}} \simeq 84.26 \text{ m}^{-1} \tag{2.29}$$

in sea water ($\sigma \simeq 4$ S/m , $\epsilon \simeq 80\epsilon_0$), the aforementioned conditions are of no practical restriction.

3. THE MODULATED SOURCE PULSE: EXACT EVALUATION OF $E_z^{lf}(\rho, t)$

Consider the current pulse

$$p(t) = \frac{1}{T_{\rm w}} \left\{ (1 - e^{-\omega_{rs} t}) u(t) - [1 - e^{-\omega_{rs}(t - T_{\rm w})}] u(t - T_{\rm w}) \right\} \sin \omega_0 t, \tag{3.1}$$

where $\omega_0 \epsilon / \sigma \ll 1$. To ensure that $p(t)$ is a continuously differentiable function of time, the width $T_{\rm w}$ and the period $T_0 = 2\pi/\omega_0$ are chosen to satisfy the relation

$$T_{\rm w}/T_0 = n/2, \quad n = 1, 2, 3, \ldots. \tag{3.2}$$

Both $T_{\rm w}$ and T_0 are assumed to be large compared to the right-hand sides of conditions (2.28). The Fourier transform of $p(t)$ is

$$\wp(\omega) = -\frac{1}{2iT_{\rm w}} [\wp^0(\omega - \omega_0) - \wp^0(\omega + \omega_0)][(-1)^n e^{i\omega T_{\rm w}} - 1] \tag{3.3a}$$

$$= -\frac{\omega_{rs}\omega_0}{iT_{\rm w}} \frac{2\omega + i\omega_{rs}}{(\omega^2 - \omega_0^2)[(\omega + i\omega_{rs})^2 - \omega_0^2]} [(-1)^n e^{i\omega T_{\rm w}} - 1], \tag{3.3b}$$

where

$$\wp^0(\omega) = \frac{1}{\omega} \frac{\omega_{rs}}{\omega + i\omega_{rs}} = \frac{1}{i} \left(\frac{1}{\omega} - \frac{1}{\omega + i\omega_{rs}} \right). \tag{3.4}$$

With Eqs. (2.23–2.26), $I_j(\rho, t)$, $j = 1$, 2, 3, read

$$I_j(\rho, t) = T_{\rm w}^{-1} [(-1)^n \hat{I}_j(\rho, t - T_{\rm w}) - \hat{I}_j(\rho, t)], \tag{3.5}$$

$$\hat{I}_1(\rho, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \omega [\wp^0(\omega - \omega_0) - \wp^0(\omega + \omega_0)] \times e^{-(1-i)a\rho\sqrt{\omega}} e^{-i\omega t}, \tag{3.6}$$

$$\hat{I}_2(\rho, t) = -\frac{1+i}{4\pi} \int_{-\infty}^{\infty} d\omega \sqrt{\omega}\, [\wp^0(\omega - \omega_0) - \wp^0(\omega + \omega_0)] \times e^{-(1-i)a\rho\sqrt{\omega}} e^{-i\omega t}, \tag{3.7}$$

$$\hat{I}_3(\rho, t) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} d\omega\, [\wp^0(\omega - \omega_0) - \wp^0(\omega + \omega_0)] \times e^{-(1-i)a\rho\sqrt{\omega}} e^{-i\omega t}. \tag{3.8}$$

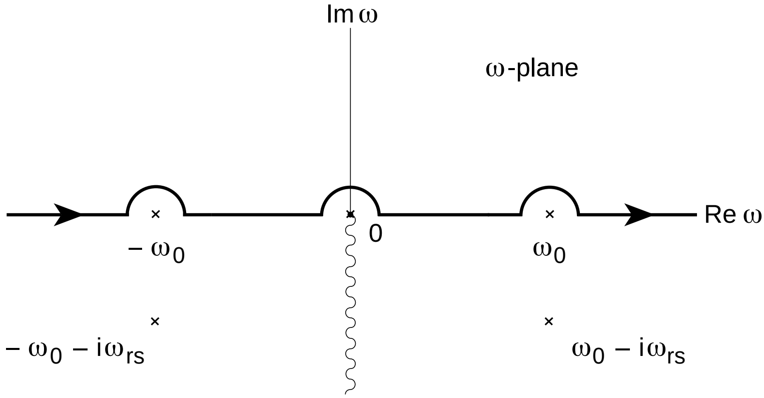

Figure 3. The original integration path and the branch-cut configuration for the integrals (3.6–3.8).

Relations (2.27) are still valid with I replaced by $\hat{I}$. Classical causality dictates that the integration path be chosen as depicted in Fig. 3.

The calculation of $\hat{I}_3(\rho, t)$ may proceed along the lines of [28]. Without further ado,

$$\hat{I}_3(\rho, t) = 0, \quad t < 0. \tag{3.9}$$

For $t > 0$, the integration path closes in the lower half of the ω-plane by taking into account the simple poles at $\omega = \pm\omega_0,\ \pm\omega_0 - i\omega_{rs}$, and wrapping the contour around the branch cut along the negative imaginary axis. The total residue contribution equals

$$\begin{aligned}\hat{I}_{3\mathrm{p}}(\rho, t) = {} & e^{-a\rho\sqrt{\omega_0}} \sin(\omega_0 t - a\rho\sqrt{\omega_0}\,) \\ & - \exp\left[-a\rho\sqrt{2}\,(\omega_0^2 + \omega_{rs}^2)^{1/4} \sin\left(\frac{\pi}{4} - \psi\right)\right] \\ & \times \sin\left[\omega_0 t - a\rho\sqrt{2}\,(\omega_0^2 + \omega_{rs}^2)^{1/4} \cos\left(\frac{\pi}{4} - \psi\right)\right] e^{-\omega_{rs} t},\end{aligned} \tag{3.10}$$

$$\psi = \frac{1}{2}\arctan\left(\frac{\omega_{rs}}{\omega_0}\right), \quad 0 < \psi < \frac{\pi}{4}. \tag{3.11}$$

The branch-cut contribution reads

$$\hat{I}_{3\mathrm{b}}(\rho, t)$$

$$= \frac{\omega_{rs}\omega_0}{\pi} \int_0^\infty d\xi\ \frac{\omega_{rs} - 2\xi}{(\xi^2 + \omega_0^2)[(\omega_{rs} - \xi)^2 + \omega_0^2]} \sin(a\rho\sqrt{2\xi}\,) e^{-\xi t} \tag{3.12a}$$

$$= \mathrm{Im}[\mathcal{I}(t; i\omega_0) - \mathcal{I}(t; \omega_{rs} + i\omega_0)], \tag{3.12b}$$

where

$$\mathcal{I}(t;\lambda) = \frac{1}{\pi}\int_0^\infty \frac{d\xi}{\xi-\lambda}\, e^{-\xi t}\sin(a\rho\sqrt{2\xi}\,), \qquad \mathrm{Im}\ \lambda > 0. \tag{3.13}$$

By virtue of

$$\begin{aligned}\int_0^\infty & d\xi\, e^{-\xi t}\sin(a\rho\sqrt{2\xi}\,) \\ &= -\sqrt{\frac{2}{t}}\ \mathrm{Re}\left[\frac{\partial}{\partial(a\rho)}\, e^{-a^2\rho^2/(2t)}\int_{-ia\rho/\sqrt{2t}}^\infty ds\, e^{-s^2}\right] \\ &= a\rho\sqrt{\frac{\pi}{2t^3}}\ e^{-a^2\rho^2/(2t)},\end{aligned} \tag{3.14}$$

$\mathcal{I}(t;\lambda)$ satisfies the differential equation

$$\frac{\partial\mathcal{I}(t;\lambda)}{\partial t} + \lambda\mathcal{I}(t;\lambda) = -a\rho(2\pi t^3)^{-1/2}e^{-a^2\rho^2/(2t)}. \tag{3.15}$$

This is readily integrated to give

$$\begin{aligned}\mathcal{I}(t;\lambda) = &\mathcal{I}(t=0^+;\lambda)e^{-\lambda t} \\ &- 2e^{-\lambda t}a\rho(2\pi t)^{-1/2}\int_1^\infty dx\ \exp\left(\frac{\lambda t}{x^2} - \frac{a^2\rho^2}{2t}\,x^2\right).\end{aligned} \tag{3.16}$$

A straightforward application of Cauchy's integral formula gives

$$\begin{aligned}\mathcal{I}(t=0^+;\lambda) &= \frac{1}{2\pi i}\left\{\int_{-\infty}^\infty dy\,\frac{y}{y^2-\lambda}\, e^{ia\rho\sqrt{2}\,y} - \int_{-\infty}^\infty dy\,\frac{y}{y^2-\lambda}\, e^{-ia\rho\sqrt{2}\,y}\right\} \\ &= e^{ia\rho\sqrt{2\lambda}}.\end{aligned} \tag{3.17}$$

In consideration of the identity

$$\begin{aligned}\exp\left(\frac{\lambda t}{x^2} - R^2x^2\right) = \frac{1}{2R}\,\frac{d}{dx}\Bigg(& e^{-2i\sqrt{\lambda t}\,R}\int_0^{Rx-i\sqrt{\lambda t}/x} ds\, e^{-s^2} \\ &+ e^{2i\sqrt{\lambda t}\,R}\int_0^{Rx+i\sqrt{\lambda t}/x} ds\, e^{-s^2}\Bigg),\end{aligned} \tag{3.18}$$

it is found that

$$\mathcal{I}(t;\lambda) = e^{ia\rho\sqrt{2\lambda}}e^{-\lambda t} - \frac{1}{2}\,e^{-R^2}\left[e^{(R+i\sqrt{\lambda t}\,)^2}\mathrm{erfc}(R+i\sqrt{\lambda t}\,) + e^{(R-i\sqrt{\lambda t}\,)^2}\mathrm{erfc}(R-i\sqrt{\lambda t}\,)\right]. \tag{3.19}$$

The square root is positive for $\lambda > 0$ and the branch cut lies in the lower λ-plane, while

$$R \equiv R(\rho,t) = \frac{a\rho}{\sqrt{2t}}, \tag{3.20}$$

as is expected by the similarity solutions of Eq. (2.21), and erfc is the complementary error function [31]

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty ds\, e^{-s^2}. \tag{3.21}$$

It follows from Eq. (3.12b) that

$$\hat{I}_{3\mathrm{b}}(\rho,t) = -\hat{I}_{3\mathrm{p}}(\rho,t) + \frac{1}{2}\,e^{-R^2}[G(Z_+) + G(Z_-) - G(Z_{0+}) - G(Z_{0-})], \tag{3.22}$$

where

$$G(Z) = \mathrm{Im}[e^{Z^2}\,\mathrm{erfc}(Z)], \tag{3.23}$$

and

$$\begin{aligned} Z_\pm \equiv Z_\pm(\rho,t) &= R \pm i\sqrt{\omega_{rs}t + i\omega_0 t} \\ &= R \mp [(\omega_0 t)^2 + (\omega_{rs}t)^2]^{1/4} e^{-i(\pi/4+\psi)}, \end{aligned} \tag{3.24a}$$

$$Z_{0\pm} \equiv Z_{0\pm}(\rho,t) = R \pm i\sqrt{i\omega_0 t} = R \mp \sqrt{\omega_0 t}\, e^{-i\pi/4}. \tag{3.24b}$$

Hence,

$$\hat{I}_3(\rho,t) = \frac{1}{2}\,e^{-R^2}[G(Z_+) + G(Z_-) - G(Z_{0+}) - G(Z_{0-})], \qquad t > 0. \tag{3.25}$$

$\hat{I}_2(\rho,t)$ and $\hat{I}_1(\rho,t)$ are obtained from Eqs. (2.27):

$$\begin{aligned} \hat{I}_2(\rho,t) = \frac{1}{2}\,e^{-R^2}\Big(&\sqrt{\omega_0}\,[F(Z_{0+}) - F(Z_{0-}) - G(Z_{0+}) + G(Z_{0-})] \\ &- \sqrt{2}\,(\omega_0^2 + \omega_{rs}^2)^{1/4}\{\cos(\pi/4-\psi)[F(Z_+) - F(Z_-)] \\ &- \sin(\pi/4-\psi)[G(Z_+) - G(Z_-)]\}\Big), \end{aligned} \tag{3.26}$$

$$\begin{aligned} \hat{I}_1(\rho,t) = e^{-R^2}\,\{&\omega_0[F(Z_{0+}) + F(Z_{0-}) - F(Z_+) - F(Z_-)] \\ &- \omega_{rs}[G(Z_+) + G(Z_-)]\}\,, \quad t > 0, \end{aligned} \tag{3.27}$$

where

$$F(Z) = \mathrm{Re}[e^{Z^2}\,\mathrm{erfc}(Z)]. \tag{3.28}$$

Note that in the limit $\omega_0 \to 0^+$ all $\hat{I}_j(\rho,t)$ tend to vanish, as they should.

A final exact expression for $E_z^{\mathrm{lf}}(\rho,t)$ easily follows from Eqs. (2.23) and (3.5):

$$E_z^{\mathrm{lf}}(\rho,t) = -\frac{\mu_0 a h_e I_0}{8\pi T_{\mathrm{w}}} \begin{cases} 0, & t \le 0, \\ e(\rho,t), & 0 < t \le T_{\mathrm{w}}, \\ e(\rho,t) - (-1)^n\, e(\rho,t-T_{\mathrm{w}}), & t > T_{\mathrm{w}}, \end{cases} \tag{3.29}$$

where

$$\begin{aligned} e(\rho,t) = \frac{e^{-R^2}}{(2t)^{3/2}} \Big\{ & 4\Omega_0^2[F(Z_{0+}) + F(Z_{0-}) - F(Z_+) - F(Z_-)] \\ & - 4\Omega_{rs}^2[G(Z_+) + G(Z_-)] \\ & + \Omega_0\sqrt{2}[F(Z_{0+}) - F(Z_{0-}) - G(Z_{0+}) + G(Z_{0-})] \\ & - 2(\Omega_0^4 + \Omega_{rs}^4)^{1/4} \Big(\cos(\pi/4 - \psi)[F(Z_+) - F(Z_-)] \\ & - \sin(\pi/4 - \psi)[G(Z_+) - G(Z_-)] \Big) \\ & + G(Z_+) + G(Z_-) - G(Z_{0+}) - G(Z_{0-}) \Big\}, \end{aligned} \tag{3.30}$$

and R, $Z_\pm$ and $Z_{0\pm}$ are defined by Eqs. (3.20) and (3.24).

4. SIMPLIFIED FORMULAS FOR $E_z^{lf}(\rho,t)$

A practically appealing case arises when $\omega_0 t \gg 1$ for $0 < t < T_{\mathrm{w}}$ ($\omega_0 T_{\mathrm{w}} \gg 1$) or $\omega_0(t - T_{\mathrm{w}}) \gg 1$ for $t > T_{\mathrm{w}}$. Instead of employing the large-argument approximation for the complementary error function in Eqs. (3.29–3.30), it is more advantageous to have recourse to the integral of Eq. (3.12a). The change of variable $\xi = R^2\bar{\xi}^2/t$ recasts this integral in the form

$$\begin{aligned} &\hat{I}_{3\mathrm{b}}(\rho,t) \\ &\quad = \frac{\bar{\Omega}_0^2\bar{\Omega}_{rs}^2}{\pi} \int_{-\infty}^{\infty} d\bar{\xi}\,\bar{\xi}\, \frac{\bar{\Omega}_{rs}^2 - 2\bar{\xi}^2}{(\bar{\xi}^4 + \bar{\Omega}_0^4)[(\bar{\Omega}_{rs}^2 - \bar{\xi}^2)^2 + \bar{\Omega}_0^4]} \sin(2R^2\bar{\xi}) e^{-R^2\bar{\xi}^2} \\ &\quad = \frac{\bar{\Omega}_0^2\bar{\Omega}_{rs}^2}{\pi} e^{-R^2}\, \mathrm{Im} \int_{-\infty}^{\infty} d\bar{\xi}\,\bar{\xi}\, \frac{\bar{\Omega}_{rs}^2 - 2\bar{\xi}^2}{(\bar{\xi}^4 + \bar{\Omega}_0^4)[(\bar{\Omega}_{rs}^2 - \bar{\xi}^2)^2 + \bar{\Omega}_0^4]} e^{-R^2(\bar{\xi}-i)^2}. \end{aligned} \tag{4.1}$$

In the above,

$$\bar{\Omega}_0 = \frac{\sqrt{\omega_0 t}}{R}, \quad \bar{\Omega}_{rs} = \frac{\sqrt{\omega_{rs} t}}{R}. \tag{4.2}$$

The integration path is shifted parallel to the real axis so that it finally passes through $\bar{\xi} = i$, as shown in Fig. 4. The condition

$$R = \frac{a\rho}{\sqrt{2t}} < [(\omega_0 t)^2 + (\omega_{rs} t)^2]^{1/4} \sin(\pi/4 - \psi), \tag{4.3}$$

along with $R < \sqrt{\omega_0 t/2}$, where ψ is defined by Eq. (3.11), ensure that the contour does not enclose any poles of the integrand. Note that the total contribution from these poles would exactly cancel $\hat{I}_{3\mathrm{p}}(\rho, t)$ of Eq. (3.10). With the change of variable $R(\bar{\xi} - i) = x$ (x: real) along the new path, $\hat{I}_{3\mathrm{b}}(\rho, t)$ reads

$$\begin{aligned} \hat{I}_{3\mathrm{b}}(\rho, t) = \frac{\bar{\Omega}_0^2 \bar{\Omega}_{rs}^2}{\pi R} e^{-R^2} \operatorname{Im} \int_{-\infty}^{\infty} dx\, (i + x/R) \\ \times \frac{\bar{\Omega}_{rs}^2 - 2(i + x/R)^2}{[(i + x/R)^4 + \bar{\Omega}_0^4]\{[\bar{\Omega}_{rs}^2 - (i + x/R)^2]^2 + \bar{\Omega}_0^4\}} e^{-x^2}. \end{aligned} \tag{4.4}$$

The contribution to integration from the region $\{x\colon |x| \gg 1\}$ is recognized as negligibly small. It follows that if

$$R^2 < O(\omega_0 t), \quad R^2 < O(\omega_{rs} t), \qquad \omega_0 t,\, \omega_{rs} t \gg 1, \tag{4.5}$$

the rational function in the integrand can be approximated as

$$\begin{aligned} \frac{\bar{\Omega}_{rs}^2 - 2(i + x/R)^2}{[(i + x/R)^4 + \bar{\Omega}_0^4]\{[\bar{\Omega}_{rs}^2 - (i + x/R)^2]^2 + \bar{\Omega}_0^4\}} &\sim \frac{\bar{\Omega}_{rs}^2}{\bar{\Omega}_0^4(\bar{\Omega}_{rs}^4 + \bar{\Omega}_0^4)} \\ &= \frac{\omega_{rs}}{\omega_0^2(\omega_0^2 + \omega_{rs}^2)} \end{aligned}$$

and pulled out of the integral. Therefore, under conditions (4.5) and (4.3), $\hat{I}_{3\mathrm{b}}(\rho, t)$ becomes

$$\begin{aligned} \hat{I}_{3\mathrm{b}}(\rho, t) &\sim \frac{\omega_{rs}^2}{\pi \omega_0 (\omega_{rs}^2 + \omega_0^2)} \int_0^{\infty} d\xi\, \sin(a\rho\sqrt{2\xi}\,) e^{-\xi t} \\ &= \frac{\omega_{rs}^2}{\omega_0^2 + \omega_{rs}^2} \frac{\beta_0 \rho}{\sqrt{2\pi}\, (\omega_0 t)^{3/2}} e^{-(\beta_0 \rho)^2/(2\omega_0 t)}, \end{aligned} \tag{4.6}$$

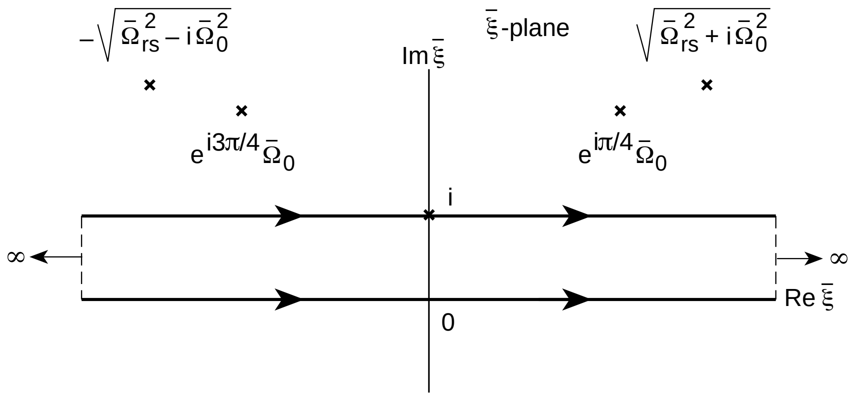

Figure 4. Contour of integration for the integral of (4.1). $e^{i\pi/4}\bar{\Omega}_0$, $\sqrt{\bar{\Omega}_{rs}^2 + i\bar{\Omega}_0^2} = (\bar{\Omega}_{rs}^4 + \bar{\Omega}_0^4)^{1/4} e^{i(\pi/4-\psi)}$, $e^{i3\pi/4}\bar{\Omega}_0$, and $-\sqrt{\bar{\Omega}_{rs}^2 - i\bar{\Omega}_0^2}$ $= (\bar{\Omega}_{rs}^4 + \bar{\Omega}_0^4)^{1/4} e^{i(3\pi/4+\psi)}$ are the poles of the integrand in the first and second quadrants ($0 < \psi < \pi/4$).

with the definition

$$\beta_0 = \beta(\omega_0) = a\sqrt{\omega_0} \sim \text{Re } k(\omega = \omega_0) \sim \text{Im } k(\omega = \omega_0). \tag{4.7}$$

From Eq. (3.10),

$$\hat{I}_{3\text{p}}(\rho, t) \sim e^{-\beta_0 \rho} \sin(\omega_0 t - \beta_0 \rho), \quad \omega_{rs} t \gg 1. \tag{4.8}$$

When $\beta_0 \rho \gg 1$, $\omega_0 t = O(\beta_0^2 \rho^2)$, and $\omega_0/\omega_{rs} \leq O(1)$,

$$|\hat{I}_{3\text{p}}(\rho, t)| = O(e^{-\beta_0 \rho}) \ll |\hat{I}_{3\text{b}}(\rho, t)| = O\left(\frac{\omega_{rs}^2}{\omega_0^2 + \omega_{rs}^2} \frac{1}{\beta_0^2 \rho^2} \right), \tag{4.9}$$

which indicates that the initial transient response of the dipole dominates the advancing signal of carrier frequency ω_0 at sufficiently large distances ρ. If $R(\rho, t) \to \infty$ and $\omega_{rs} t$, $\omega_0 t$ are kept fixed, conditions (4.3) and (4.5) are violated. However, since the pole contributions to the value of $\hat{I}_{3\text{b}}(\rho, t)$ exactly cancel $\hat{C}_{3\text{p}}(\rho, t)$, the initial transient response prevails.

$\hat{I}_3(\rho,t)$ is expressed as

$$\begin{aligned}\hat{I}_3(\rho,t) &= \hat{I}_{3\mathrm{p}}(\rho,t) + \hat{I}_{3\mathrm{b}}(\rho,t)\\ &\sim e^{-\beta_0\rho}\sin(\omega_0 t - \beta_0\rho) + e^{-a^2\rho^2/(2t)}\,\frac{\omega_{rs}^2 a\rho}{\omega_0(\omega_{rs}^2+\omega_0^2)}\,(2\pi t^3)^{-1/2}.\end{aligned} \tag{4.10}$$

Note that the effect of the rise time $\tau_{rs} = \omega_{rs}^{-1}$ is the decrease of the transient amplitude by the factor $\omega_{rs}^2/(\omega_{rs}^2+\omega_0^2)$. $\hat{I}_2(\rho,t)$ and $\hat{I}_1(\rho,t)$ are approximated by

$$\begin{aligned}\hat{I}_2(\rho,t) &\sim \sqrt{\omega_0}\,e^{-\beta_0\rho}\,[\sin(\omega_0 t - \beta_0\rho) + \cos(\omega_0 t - \beta_0\rho)]\\ &\quad + e^{-a^2\rho^2/(2t)}\,\frac{\omega_{rs}^2}{\omega_0(\omega_{rs}^2+\omega_0^2)}\,(2\pi t^3)^{-1/2}\left(\frac{a^2\rho^2}{t}-1\right),\end{aligned} \tag{4.11}$$

$$\begin{aligned}\hat{I}_1(\rho,t) &\sim 2\omega_0 e^{-\beta_0\rho}\,\cos(\omega_0 t - \beta_0\rho)\\ &\quad + e^{-a^2\rho^2/(2t)}\,\frac{\omega_{rs}^2 a\rho}{\omega_0(\omega_{rs}^2+\omega_0^2)}\,(2\pi t^5)^{-1/2}\left(\frac{a^2\rho^2}{t}-3\right).\end{aligned} \tag{4.12}$$

The corresponding approximate formula for $E_z^{\mathrm{lf}}(\rho,t)$ is

$$E_z^{\mathrm{lf}}(\rho,t) \sim -\frac{\mu_0 h_e I_0 \beta_0 \omega_0}{4\pi T_\mathrm{w}} \times \begin{cases} 0, & t \le 0,\\ e^{-\beta_0\rho}\left\{\left[\dfrac{1}{(\beta_0\rho)^3}+\dfrac{1}{(\beta_0\rho)^2}\right]\sin(\omega_0 t-\beta_0\rho) \right.\\ \quad \left. + \left[\dfrac{1}{(\beta_0\rho)^2}+\dfrac{2}{\beta_0\rho}\right]\cos(\omega_0 t-\beta_0\rho)\right\}\\ \quad + \dfrac{2\omega_{rs}^2}{\omega_0^2+\omega_{rs}^2}\,\dfrac{1}{\sqrt{2\pi(\omega_0 t)^5}}\left[\dfrac{(\beta_0\rho)^2}{2\omega_0 t}-1\right]e^{-(\beta_0\rho)^2/(2\omega_0 t)}, & \\ & 0 < t \le T_\mathrm{w},\ \omega_{0,rs}t \gg 1,\\ \dfrac{2\omega_{rs}^2}{\omega_0^2+\omega_{rs}^2}\left\{\dfrac{1}{\sqrt{2\pi(\omega_0 t)^5}}\left[\dfrac{(\beta_0\rho)^2}{2\omega_0 t}-1\right]e^{-(\beta_0\rho)^2/(2\omega_0 t)}\right.\\ \quad + (-1)^{n+1}\dfrac{1}{\sqrt{2\pi[\omega_0(t-T_\mathrm{w})]^5}}\,e^{-(\beta_0\rho)^2/[2\omega_0(t-T_\mathrm{w})]}\\ \quad \left.\times\left[\dfrac{(\beta_0\rho)^2}{2\omega_0(t-T_\mathrm{w})}-1\right]\right\}, & \omega_{0,rs}(t-T_\mathrm{w}) \gg 1.\end{cases} \tag{4.13}$$

Note that the sinusoidal signal is exactly canceled for $t > T_\mathrm{w}$.

The formula for $E_z^{\mathrm{lf}}(\rho,t)$ is simplified considerably when $\beta_0\rho \ll 1$:

$$E_z^{\mathrm{lf}}(\rho,t) \sim -\frac{\mu_0 h_e I_0 \beta_0 \omega_0}{4\pi T_{\mathrm{w}}} \times \begin{cases} 0, & t \le 0, \\ \dfrac{1}{(\beta_0\rho)^3}\,\sin(\omega_0 t - \beta_0\rho) \\ \quad -\dfrac{2\omega_{rs}^2}{\omega_0^2+\omega_{rs}^2}\,\dfrac{1}{\sqrt{2\pi(\omega_0 t)^5}}, & 0 < t \le T_{\mathrm{w}}, \quad \omega_{0,rs} t \gg 1, \\ -\dfrac{2\omega_{rs}^2}{\omega_0^2+\omega_{rs}^2}\,\dfrac{1}{\sqrt{2\pi}}\left[\dfrac{1}{\sqrt{(\omega_0 t)^5}} \right. \\ \quad \left. +\dfrac{(-1)^{n+1}}{\sqrt{\omega_0^5 (t-T_{\mathrm{w}})^5}}\right], & \omega_{0,rs}(t-T_{\mathrm{w}}) \gg 1. \end{cases} \tag{4.14}$$

Note that the field close to the dipole is essentially an out-of-phase replica of the current source pulse, as expected by inspection of Eq. (2.22).

When $R(\rho,t) \gg 1$ and $R(\rho,t-T_{\mathrm{w}}) \gg 1$ for $t > T_{\mathrm{w}}$, $E_z^{\mathrm{lf}}(\rho,t)$ becomes

$$E_z^{\mathrm{lf}}(\rho,t) \sim -\frac{\mu_0 h_e I_0 \beta_0 \omega_0}{4\pi T_{\mathrm{w}}} \times \begin{cases} 0, & t \le 0, \\ e^{-\beta_0\rho}\,\dfrac{2}{\beta_0\rho}\,\cos(\omega_0 t - \beta_0\rho) \\ \quad +\dfrac{\omega_{rs}^2}{\omega_0^2+\omega_{rs}^2}\,\dfrac{e^{-a^2\rho^2/(2t)}}{\sqrt{2\pi(\omega_0 t)^5}}\,\dfrac{a^2\rho^2}{t}, & \begin{cases} R(\rho,t) \gg 1, \\ 0 < t \le T_{\mathrm{w}}, \end{cases} \\ \dfrac{\omega_{rs}^2}{\omega_0^2+\omega_{rs}^2}\left[\dfrac{e^{-a^2\rho^2/(2t)}}{\sqrt{2\pi(\omega_0 t)^5}}\,\dfrac{a^2\rho^2}{t} \right. \\ \quad \left. +(-1)^{n+1}\,\dfrac{e^{-a^2\rho^2/[2(t-T_{\mathrm{w}})]}}{\sqrt{2\pi[\omega_0(t-T_{\mathrm{w}})]^5}}\,\dfrac{a^2\rho^2}{t-T_{\mathrm{w}}}\right], & \begin{cases} R(\rho,t-T_{\mathrm{w}}) \gg 1, \\ t > T_{\mathrm{w}}. \end{cases} \end{cases} \tag{4.15}$$

According to formulas (4.14) and (4.15), the field for $0 < t < T_{\mathrm{w}}$ shifts from the form of the original source pulse when $\beta_0\rho \ll 1$ to the form of its time derivative when $R(\rho,t) \gg 1$.

Evidently, the condition $\omega_{rs} \gg \omega_0$ and the subsequent approximation

$$\frac{\omega_{rs}^2}{\omega_0^2+\omega_{rs}^2} \sim 1$$

reduce formula (4.13) to that for the low-frequency field generated by the rectangular wave packet [26]

$$p(t) = \frac{1}{T_{\mathrm{w}}}\left[u(t) - u(t - T_{\mathrm{w}})\right]\sin\omega_0 t.$$

5. DISCUSSION AND CONCLUSION

The present paper concludes previous work by the same author on the propagation of electromagnetic pulses excited by exponential currents with finite rise and decay times [28]. The electric field created by a Hertzian dipole in sea water is evaluated analytically under the low-frequency approximation, when the source current is a *modulated* exponential pulse. The fact that the requisite integrals can be carried out *exactly* depends crucially on the Fourier transform $\wp(\omega)$ of the excitation pulse $p(t)$ being a meromorphic function of ω . The results obtained hitherto can be extended by mere inspection to many-parameter excitations that ensue from acting successively on the pulse of Eq. (1.2) with conventional low-pass filters (for instance, resistor-capacitor circuits) and modulating the output signal. In view of Mittag-Leffler's theorem [32], any signal that is constructed by prescribing the location of poles and the singular parts of $\wp(\omega)$ at those poles admits a low-frequency response as a suitable combination of real and imaginary parts of the complementary error function with complex arguments.

The sinusoidal signal is superimposed to the original transient, shifting successively from the form of the excitation current when $R(\rho,t) \ll 1$ to its time derivative when $R(\rho,t) \gg 1$ and the transients tend to dominate the advancing wavepacket. The finite rise and decay time $\tau_{rs} = \omega_{rs}^{-1}$ modifies the amplitude of the transient by the multiplicative factor $\omega_{rs}^2(\omega_0^2 + \omega_{rs}^2)^{-1}$. Because the exposition is based on the simplifying condition $\omega\epsilon/\sigma \ll 1$ in the frequency domain, the analytical results here are by no means restrictive to sea water but can also be applied to any other highly conducting medium.

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