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WAVE PACKETS AND GROUP VELOCITY IN ABSORBING MEDIA: SOLUTIONS OF THE TELEGRAPHER'S EQUATION

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1. PRELIMINARY REMARKS

The problem of propagation of wave packets (henceforth WP, WPs, as required) in an absorbing medium was addressed for the first time by Sommerfeld [1], and Brillouin [2, 3]. They studied the propagation of light in absorbing dielectrics within the framework of the Lorentz model and analyzed the behavior of the group velocity defined as $U = (dk_r/d\omega)^{-1}$ for real frequencies ω and for complex wave numbers $k = k_r + ik_i$. They showed that the group velocity U may exceed the speed of light c and even become infinite or negative in the absorption range of frequencies. It was also shown that for a pulse that is switched on instantaneously at the boundary and then becomes sinusoidal with a constant amplitude, there is a precursor wave propagating with the speed of light c, while the main body of the pulse propagates with a "signal velocity" S, significantly differing from U in the absorption frequency range, where the wave dispersion is anomalous. The signal velocity S was found to be positive everywhere and smaller than the light speed c. A detailed asymptotic analysis of the Sommerfeld-Brillouin problem and numerical calculations of the signal velocity for the Lorentz medium were carried out in [4, 5].

During several decades after the appearance of Sommerfeld and Brillouin's classical works, the cases of abnormal group velocity ($U < 0$ or $U > c$) were regarded as nonphysical. However, the theoretical study of Garrett and McCumber [6] and the experiment of Chu and Wong [7] availed new insight into the role of abnormal group velocity. Instead of a unit-step modulated signal considered in [1–3], Garrett and McCumber analyzed the propagation of a Gaussian-envelope modulated signal through a slab region of a medium having an absorption line near the central frequency of the pulse. They have shown that for a sufficiently thin slab, the envelope's maximum propagates with abnormal group velocity. In particular, it was shown that the peak of the pulse emerges at the instant predicted by the group velocity U even if U exceeds c or becomes negative. Such a behavior is associated with the reshaping which a pulse undergoes in a dispersive absorbing medium, and does not necessarily imply a violation of the special relativity theory and its causation considerations. In other words, due to the attenuation, various parts of the pulse might be differently affected, thus leading to reshaping of the pulse envelope. Consequently, in addition to propagation effects, the location of the peak of the pulse might appear at a different location. When this is construed as actual motion of the envelope, for finite traveling distances speeds in excess of the light speed may result.

Chu and Wong [7] measured the pulse velocity in samples of $GaP:N$ with a tunable laser and found that the measured values agree satisfactorily with the calculated group velocity, including the frequency range where U is abnormal and, in particular, superluminal. Thus, their results have confirmed the predictions of Garrett and McCumber [6]. An asymptotic analysis of Gaussian WPs in a Lorentz gas has been performed in [8], where it was shown that the fast propagation with $U > c$ observed by Chu and Wong is characteristic for the early stage of the motion of the WP, while the slow propagation with $U < c$ appears at long traveling distances.

The purpose of the present paper is to investigate the propagation of WPs in absorbing media, for which the process of wave propagation is governed by the telegrapher's equation (henceforth TE), both for WPs containing many oscillations, and for short single burst pulses, henceforth referred to as solitary pulses, abbreviated SPs. Here we aim to investigate the evolution of causal WPs that are emitted from the boundary $x = 0$ into the domain $x > 0$, within the framework of the TE. In particular, we aim to understand where and when superluminal and subluminal values of the group velocity appear as relevant kinematic characteristics of WPs.

The question of WPs and their associated group velocities is not exclusively of academic interest. Rather, there are important application-oriented aspects. Every application of wave propagation to distance measurement, be it in ionospheric exploration involving ionosonde apparatus or geophysical applications involving scattering of elastodynamic waves from underground structures, essentially is based on measurement of time of arrival of pulses (in some cases, e.g., chirp radar equipment, the signals are continuous and frequency modulated, but the end effect is the same when the continuous signals are post-processed in the receiver into short bursts). Thus knowing the pulse velocity in the medium at hand, measurement of time of arrival facilitates the assessment of distances.

2. INTRODUCTION AND BACKGROUND

The TE is a one dimensional wave equation given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda}\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where henceforth c is identified as the characteristic velocity which is equal to the phase velocity in the absence of damping or amplification which are represented by the first order time-derivative term.

Numerous examples of waves in various absorbing media described by (1) are presented in the literature. Several applications of the TE to physical systems are mentioned below. Thus, for electromagnetic waves in conducting media (semiconductors or dense, slightly ionized plasmas, for which the electron collision frequency is much larger than the plasma frequency), the parameters entering into (1) are given by

$$c = c_0\ ,\ 1/\lambda = \sigma/\varepsilon_0 \tag{2}$$

where c_0 is the vacuum speed of light, σ is the electrical conductivity and ε_0 is the permittivity of free space. For a coaxial cable line characterized by unit-length parameters: capacitance C, inductance L, and resistance R, and with an ideal insulation between the coaxial electrodes (i.e., zero shunt conductance), (1) is obtained for the interelectrode voltage, and its parameters are (see for example Stratton [9], p.550)

$$c = 1/\sqrt{LC} = c_0/\sqrt{\varepsilon}\ ,\ 1/\lambda = R/L \tag{3}$$

where ε is the relative dielectric permittivity of the interelectrode spacer. For acoustic waves propagating in a slightly ionized gas in the presence of an external magnetic field B, in the direction perpendicular to the vector B, (1) describes small pressure perturbations [10]. For this case we have

$$c = \sqrt{(\partial\rho/\partial p)_s}\ ,\ 1/\lambda = \sigma B^2/\rho \tag{4}$$

where c is the sound velocity determined by the isentropic derivative of the gas pressure p with respect to its density ρ, and σ is the electrical conductivity of the gas.

Beyond possible applications of the WP solutions of the TE to communication and radar-type distance measurement, our analysis was also motivated by the behavior of the classical group velocity

$$W(k) = \partial\omega/\partial k \tag{5}$$

in the presence of damping effects, and the analysis of WP solutions of (1) provides an appropriate arena. In the present case the dispersion equation becomes complex, hence the classical group velocity $W(k) = \partial\omega/\partial k$ becomes complex too, in general. Inasmuch as the group velocity is associated with the flux of energy through space, the conceptual ramifications must be re-examined. The dispersion relation associated with (1) is given below. The associated quantity $W(k)$ is real in two cases: (a) when the wave number k is purely imaginary, and (b) when k is real and larger than $k_c = 1/(2c\lambda) = 1/L_a$, where L_a is the characteristic absorption length. In the case (b) the group velocity is larger than the light speed c. The present paper continues to study the role of the real group velocity in absorbing systems [11] (see also [11] for earlier references). Our choice of the TE for the analysis of WPs stems also from the fact that the TE is a second-order equation of hyperbolic type, for which the solution of initial-, and boundary-value problems for arbitrary signals propagating in a half-space can be presented in a closed integral form in terms of modified Bessel functions and the value of the signal at the boundary [12]. This allows us, on one hand, to ensure the causality of the problem of pulse propagation, and on the other hand, to avoid cumbersome direct calculations of the Fourier integrals for short and moderate traveling distances.

It should be noted that for WP solutions obtained in [6, 8] for the Lorentz model, the group velocity U was calculated by

$$U = [(\partial k_r/\partial\omega)_{\omega=\omega_c}]^{-1} \tag{6}$$

where ω_c is the carrier frequency. For the Lorentz model, the group velocity W defined in (5) is complex for real ω. For short traveling distances the velocity of the envelope's maximum $V(t)$ was shown to be close to U [6, 8], while for long distances V significantly differs from U. Similar results were obtained in [11, 13] for whistler WPs propagating in a magnetized collisional plasma, for which the form of complex dispersion equation is quite different from the one following from the Lorentz model. For large distances the Gaussian envelope's peak of a whistler WP is decelerated and its velocity $V(t)$ may become much less than U. As was shown in [11], for large times and distances, any small section of a WP can be characterized by a local complex frequency $\tilde{\omega}(x,t)$ and by a local complex wave number $\tilde{k}(x,t)$. The quantities $\tilde{\omega}$ and $\tilde{k}$ are connected through a complex dispersion equation for an absorbing medium, while $\tilde{\omega}$ represents the saddle point of the eikonal $\omega t - k(\omega)x$ in the complex plane ω.

Asymptotically both quantities $\tilde{\omega}$ and $\tilde{k}$ depend only on the ratio $x/t = W$, where W is the classical group velocity (5) calculated at $k = \tilde{k}$. The asymptotic group velocity W is always real and it represents the velocity of propagation of constant complex values of the local frequency $\tilde{\omega}$ and the local wave number $\tilde{k}$. Since in an absorbing medium the local frequency $\tilde{\omega}$ at the center of a WP changes in time, the envelope's peak propagates with a variable velocity $V(t)$. The results of [11] related to asymptotic stage of WP propagation in absorbing media are quite general and, therefore, the real group velocity should appear at long times in solutions of the TE.

Thus, we may expect that for causal WP solutions of TE, the envelope's peak velocity V will be close to the quantity $W > c$ during some finite time interval and that the subluminal real group velocity $W = x/t < c$ will be revealed for WP solutions of TE at long traveling distances.

The paper is organized as follows: In Sec. 3 the classical group velocity $W(k) = \partial\omega/\partial k$ is calculated for elementary wave solutions of TE. It is shown that the group velocity is real for real wave numbers k if the absolute value of k exceeds some threshold value and in that case $W > c$. The quantity W is real also for purely imaginary values of k and in this case $W < c$. In addition, the group velocity W_c defined in (5) is calculated for real carrier frequency $\omega = \omega_c$. The quantity W_c is complex and its real part U_c is much larger than the imaginary part of W_c in the range of frequencies $\omega > 1/\lambda$. It is shown that in this range $U_c > c$ and U_c tends to c when ω tends to infinity. The solution $u(x,t)$ of TE in a half-space $x > 0$ satisfying zero initial conditions and a causal boundary condition $u(0,t) = F(t)$ with $F = 0$ for $t \leq 0$ is presented in the form of the Fourier integral and its asymptotic representation is obtained by the saddle-point method. It is shown that for long times $(t - x/c \gg \lambda)$ and large distances $(x \gg c\lambda)$, the saddle-point frequency $\tilde{\omega}$ is purely imaginary and the solution becomes aperiodic in time even if the boundary signal $F(t)$ contains oscillations. In this limit the quantity $\tilde{\omega}$ depends on the ratio x/t and represents the logarithmic slope of the SP solution as a function of time. A given constant value of $\tilde{\omega}$ propagates with the real group velocity $W = x/t < c$. Analysis of the asymptotic behavior of the Green function for TE shows that in any fixed spatial domain behind the leading edge $x = ct$ the solution of TE approaches at $t \gg \lambda$ the solution of the diffusion equation that can be obtained by discarding the second time derivative of (1).

In Sec. 4 we analyze the propagation of WPs filled with high-frequency oscillations at relatively short distances satisfying the condition $x \ll c\lambda^2/\Delta$, where Δ is the characteristic semiwidth of the boundary signal envelope, which is assumed to be less than the damping time λ. It is shown that for such short distances the carrier frequency ω remains practically constant and the maximum of the envelope propagates with the group velocity that is larger than c. Since this happens only at the initial stage of WP propagation, there is no violation of special relativity: The envelope's maximum appears at the spatial point x only after the instant x/c when the light precursor reaches this point and, hence, the maximum never overtakes the leading edge of the signal.

Section 5 contains the analysis of propagation of SPs, at distances x which are much smaller than the absorption length $L_a = 2c\lambda$. It is assumed that the characteristic lifetime of a pulse given at the boundary $x = 0$ is much shorter than the damping time λ. It is shown that the maximum of a SP propagates with the velocity V, which is smaller than c, and V tends to c when the lifetime of a boundary signal tends to zero. In the analysis of both problems considered in Sects. 4 and 5, the integral representation of the solution of the TE based on the explicit form of the Green function has been employed.

In Section 6 is shown how the results related to short distance propagation can be transferred to the case of the generalized telegrapher's equation (henceforth GTE), which is obtained by the addition of a dispersive term $\omega_0^2 u$ to the left-hand side of (1).

Section 7 presents the results of numerical calculations of high-frequency signals and SPs. Both kinds of signals are computed directly from the integral representation of the solution of the TE given in [12]. The calculations of high-frequency signals demonstrate the superluminal propagation of a temporal envelope maximum $(V > c)$ at short distances $x \ll L_a$ with the transition to the subluminal mode of propagation for longer distances. The numerical calculations are performed also for relatively large distances (of order of L_a) that are beyond the validity of the asymptotic method of Sec. 4.

Numerical calculations of SPs presented in Sec. 7 demonstrate the subluminal propagation of their temporal maxima $(V < c)$. In addition, it is shown that at the distance x which is close to five absorption lengths, a new temporal maximum is created and this results in a significant reshaping of the pulse. For larger distances the new maximum dominates and propagates with $V < c$, while the magnitude of the solution at the initial maximum decreases and the latter totally disappears at $x > 10\ L_a$.

The discussion of main results is given in Sec. 8.

3. GROUP VELOCITY, LONG-TIME ASYMPTOTICS, AND THE GREEN FUNCTION

The elementary exponential solutions of (1) have the form

$$u = C\ \exp[i(kx - \omega(k)t)] \tag{7}$$

where the dispersion relation $\omega(k)$ is given by

$$\omega(k) = -\frac{i}{2\lambda} \pm \sqrt{c^2k^2 - \frac{1}{4\lambda^2}} \tag{8}$$

The upper and lower signs in (8) correspond to the branches of the function $\omega(k)$ that are responsible for the wave propagation in the positive and negative x-directions, respectively. Calculating the group velocity from (8) results in

$$\frac{\partial\omega}{\partial k} = \pm W(k)\ ,\ W(k) = \frac{c}{\sqrt{1 - 1/(4c^2k^2\lambda^2)}} \tag{9}$$

As mentioned above and seen from (9) there are two domains of possible values of k^2 where the group velocity is real:

$$k^2 > k_c^2 = (4\lambda^2c^2)^{-1}\ ,\ W > c \tag{10}$$

and

$$k^2 < 0\ ,\ 0 < W < c \tag{11}$$

Thus, the real group velocity W appears either for sufficiently short oscillatory waves satisfying condition (10) and then it exceeds the vacuum speed of light, or for the waves that satisfy condition (11) and, therefore, decay aperiodically both in space and in time. For the latter the group velocity is less than c. Deferring the discussion of the condition (10) and its role in the propagation of real signals to subsequent sections of the present paper, we show below that the case (11) is revealed at the asymptotic stage of the solution describing a SP for long times t.

Calculating the group velocity W for a given real frequency $\omega = \omega_c$ in the accordance with the definition (5) results in a complex function $W_c(\omega)$ of a real variable ω,

$$W_c(\omega) = U_c(\omega) + iV_c(\omega) \tag{12}$$

whose real and imaginary parts are presented as

$$U_c = c\frac{\sqrt{2}(\zeta+1)^{3/2}}{\zeta^2+3}, \; V_c = -c\frac{\sqrt{2}(\zeta-1)^{3/2}}{\zeta^2+3}, \; \zeta = \sqrt{1+\frac{1}{\lambda^2\omega^2}} \tag{13}$$

Analysis of formulas (13) shows that U_c reaches its maximal value $maxU_c = 1.066c$ at $\lambda\omega = \lambda\omega_m = (16-4\sqrt{13})^{-1/2} = 0.796$ and for $\omega > \omega_m$ the quantity U_c remains larger than speed of light c and tends to c at $\omega \to \infty$. Functions $U_c(\lambda\omega)$ and $V_c(\lambda\omega)$ are shown in Figs. 1 and 2. Although U_c is different from the quantity U defined in (6), both functions behave similarly. The maximal value of U equals $1.089c$ and it is reached at $\lambda\omega = 0.577$.

Let us consider the evolution of the TE for a signal excited at the boundary $x = 0$ of the half-space $x > 0$. The mathematical problem is formulated as follows: To find the solution $u(x,t)$ of (1) in the domain $0 < x < \infty$, $t > 0$ satisfying conditions:

$$u(x,0) = 0 \;, \; (\partial u/\partial t)_{t=0} = 0 \;; \; u(0,t) = F(t) \;, \; F(0) = 0 \tag{14}$$

The first two conditions (14) are satisfied automatically for causal boundary signals $F(t)$ with $F = 0$ for $t < 0$. The function $F(t)$ may represent a train of oscillations within some envelope or a solitary pulse. It is assumed that $F(t)$ tends to zero when t tends to infinity. The solution $u(x,t)$ can be presented in the form of the Fourier integral

$$u(x,t) = \int_{-\infty}^{\infty} B(\omega) \; \exp[i\Sigma(\omega,x,t)]d\omega \tag{15}$$

where $B(\omega)$ is the Fourier transform of the boundary signal

$$B(\omega) = \frac{1}{2\pi}\int_{-0}^{\infty} F(t) \; \exp(i\omega t)dt \tag{16}$$

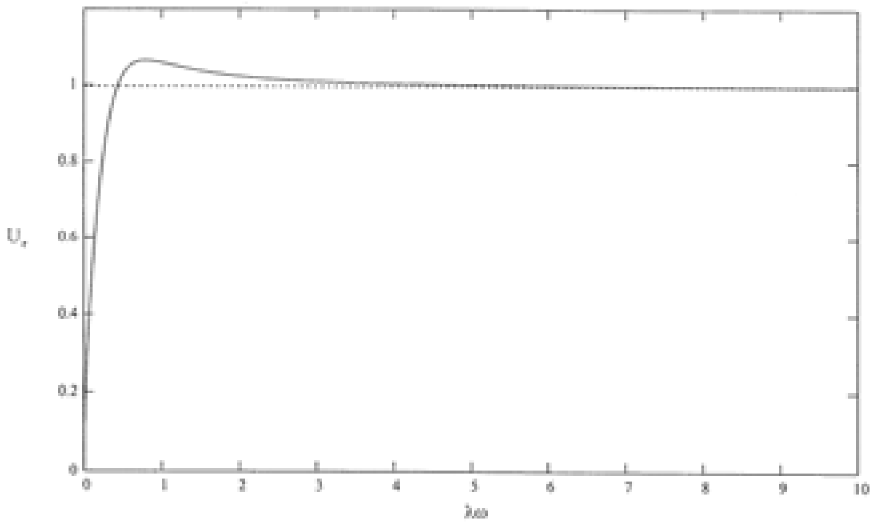

Figure 1. The real part U_c of the group velocity normalized to the speed of light c as function of the parameter $\lambda\omega$ for real frequencies ω.

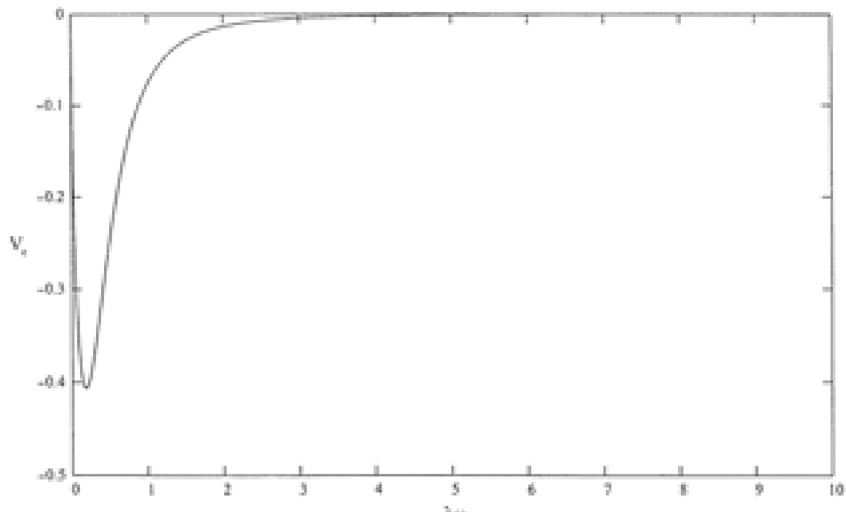

Figure 2. The imaginary part V_c of the group velocity normalized to the speed of light c as function of the parameter $\lambda\omega$ for real frequencies ω.

and Σ is the eikonal given by

$$\Sigma(\omega, x, t) = k(\omega)x - \omega t \ , \ k(\omega) = c^{-1}\sqrt{\omega(\omega + i/\lambda)} \tag{17}$$

The solution $u(x,t)$ is nonzero only in the domain $0 < x < ct$. The fact that $u = 0$ for $x > ct$ stems from hyperbolicity of (1). This can be shown also directly from the integral representation (15) by introducing finite limits of integration $-\Omega < \omega < \Omega$, applying the Jordan lemma and transition to the limit $\Omega \to \infty$.

We assume that the boundary signal $F(t)$ decays at $t \to \infty$. Therefore, its spectrum $B(\omega)$ does not have singular points on the real axis such as simple poles that appear in the Sommerfeld-Brillouin problem [1, 2]. Then employing the saddle-point method for calculating the integral given by (15) for large values of t and $x < ct$ (see Ref. [11] for details) allows us to determine the saddle point $\omega = \tilde{\omega}$ in the complex plane ω as

$$Re\left(\frac{\partial k}{\partial \omega}\right)_{\omega=\tilde{\omega}} = \frac{t}{x} \ , \ Im\left(\frac{\partial k}{\partial \omega}\right)_{\omega=\tilde{\omega}} = 0 \tag{18}$$

The second of Eqs. (18) indicates that for the saddle-point frequency $\omega = \tilde{\omega}$ the group velocity W is real and equals x/t. Since $x < ct$, the inequality $W < c$ holds, i.e., the condition (11) is satisfied. From Eqs. (8), (9) and (18) we can find the values of $\tilde{\omega}$ and $\tilde{k} = k(\tilde{\omega})$

$$\tilde{\omega} = \frac{i}{2\lambda}\left(\frac{1}{\sqrt{1 - W^2/c^2}} - 1\right) \ , \ \tilde{k} = \frac{iW/c}{2\lambda c\sqrt{1 - W^2/c^2}} \ , \ W = \frac{x}{t} \tag{19}$$

If the function $B(\omega)$ does not have singular points with $Im\omega \geq 0$, the asymptotic expression for the Fourier integral (15) obtained by the saddle-point method is as follows:

$$u(x,t) \approx \tilde{G}(x,t)B(\tilde{\omega}(x,t)) \ , \ \tilde{G}(x,t) = \frac{\exp\left[i\Sigma(\tilde{\omega}(x,t), x, t) + \dfrac{i\pi}{4}\right]}{\sqrt{2\pi x(\partial^2 k/\partial \omega^2)_{\omega=\tilde{\omega}}}} \tag{20}$$

Formula (20) is valid for $t - x/c \gg \lambda$ and $x \gg c\lambda$. Here $\tilde{G}(x,t)$ is the asymptotic expression for the Green function $G(x,t)$. The latter is defined as the solution of the initial-boundary value problem (14) with $F(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta-function. Therefore,

$G(x,t)$ is given by the Fourier integral (15) with $B = 1/2\pi$. Since in the limit $t \to \infty$ the saddle-point frequency $\tilde{\omega}$ tends to zero, $B(\tilde{\omega})$ tends to a constant value $B(0)$ which can be easily calculated from (16). As is seen from (20), for sufficiently long times and $B(0) \neq 0$ the detailed structure of the boundary signal $F(t)$ does not influence the asymptotic behavior of the solution $u(x,t)$, which will be totally determined by the Green function and by the integral of $F(t)$ over the whole time interval $0 < t < \infty$. For the TE (1) the Green function is well known and the solution of the initial-boundary value problem (14) can be found as a convolution of F and G:

$$u(x,t) = \int\limits_{x/c}^{t} F(t-\tau)G(x,\tau)d\tau \tag{21}$$

For the TE (1), the specific form of (21) can be found in Doetsch's monograph ([12], formula 25.23). Recasting the Doetsch formula in our nomenclature results in

$$u(x,t) = e^{-x/2\lambda c}F(t-x/c) + \frac{x}{2\lambda c}\int\limits_{x/c}^{t} e^{-\tau/2\lambda}\frac{I_1\left[(2\lambda)^{-1}\sqrt{\tau^2 - x^2/c^2}\right]}{\sqrt{\tau^2 - x^2/c^2}}F(t-\tau)d\tau \tag{22}$$

where $I_1(z)$ is the modified Bessel function of the first order. Since for $F(t) = \delta(t)$ the solution (22) turns into the Green function, the latter is presented as

$$\begin{aligned} G(x,t) &= 0,\ (t < x/c); \\ G(x,t) &= G_1(x,t) + G_2(x,t),\ (t > x/c) \\ G_1 &= \delta(t - x/c)e^{-x/2\lambda c}, \\ G_2 &= \frac{x}{2\lambda c}e^{-t/2\lambda}\frac{I_1\left[(2\lambda)^{-1}\sqrt{(t^2 - x^2/c^2)}\right]}{\sqrt{t^2 - x^2/c^2}} \end{aligned} \tag{23}$$

The first term G_1 on the right-hand side of (23) represents an exponentially decaying singular signal propagating with the speed of light. The second term G_2 is a regular part of the Green function that is responsible for the phenomenon which is known in mathematical physics as wave diffusion [14]: If the boundary signal has a finite lifetime, i.e., $F(t) = 0$ for $t < 0$ and for $t > T$, the signal at the point x becomes nonzero at $t = x/c$ but it does not vanish again at $t = x/c + T$. Instead, a residual perturbation persists at the point x for all times $t > x/c$ and decays as $t \to \infty$ (see also [9]). At the leading edge of the signal $(x = ct)$ the function G_2 is given by

$$G_2(x = ct, t) = \frac{te^{-t/2\lambda}}{8\lambda^2} \tag{24}$$

For large times $(t - x/c \gg \lambda)$ and long distances $(x \gg c\lambda)$ the argument of the function I_1 is large and we may replace I_1 by its asymptotic expansion

$$I_1(z) \approx e^z/\sqrt{2\pi z}\ ,\ (z \gg 1) \tag{25}$$

Then we obtain the asymptotic form of the function G_2

$$G_2(x,t) \approx \frac{x\ \exp\left(\left[\sqrt{t^2 - x^2/c^2} - t\right]/2\lambda\right)}{c\sqrt{4\pi\lambda}(t^2 - x^2/c^2)^{3/4}}\ ,\ [t - x/c \gg \lambda,\ x \gg c\lambda] \tag{26}$$

The same asymptotic formula is obtained from the saddle-point representation of the Green function $\tilde{G}(x,t)$ given by the second equation in (20), in which $\tilde{\omega}$ is taken from the first equation in (19). For $t \gg \lambda$ the exponential term in (26) represents a fast varying function which is equal to $\exp(i\tilde{\Sigma})$, where $\tilde{\Sigma}$ is the value of the eikonal $\Sigma = k(\omega)x - \omega t$ calculated at the saddle point $\omega = \tilde{\omega}(x,t)$. Both quantities $\tilde{\omega}$ and $\tilde{k} = k(\tilde{\omega})$ are purely imaginary, therefore, $i\tilde{\Sigma}$ is real and consequently the signal (23) displays an aperiodic behavior. This is quite different from a regular WP comprising a large number of oscillations. As is seen from (26), for large distances and for a fixed ratio $x/t = W < c$ the Green function can be presented as $G_2 = a(W)t^{-1/2}\exp[-b(W)t]$ with $b = i(\tilde{\omega} - \tilde{k}W) > 0$.

Therefore, the velocity of propagation of a given damping rate b equals $W = x/t < c$ and this is the real group velocity $\partial\omega/\partial k$ calculated for the imaginary wave number $\tilde{k}$. For a fixed value of x the pulse's slope and the velocity of its propagation tend to zero when $t \to \infty$. In the limit $W^2/c^2 \ll 1$ (26) takes the form

$$G_2 \approx \frac{xe^{-x^2/4\nu t}}{2\sqrt{\pi\nu}t^{3/2}}\ ,\ \nu = c^2\lambda\ ,\ (x/ct)^2 \ll 1 \tag{27}$$

It can be easily shown that the function on the right-hand side of (27) represents the Green function of the initial boundary value problem for the diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{28}$$

This result is quite natural: For long times the solution $u = G_2$ of (1) satisfies the condition

$$\left| \lambda(\partial^2 u/\partial t^2)/(\partial u/\partial t) \right| = O\left[\lambda t/(t^2 - x^2/c^2)\right] \ll 1 \tag{29}$$

which indicates that for $t - x/c \gg \lambda$ the first term on the left-hand side of (1) can be neglected. Therefore, in this limit the asymptotic behavior of the solution $u(x,t)$ of the TE can be described by (28). For a given x the temporal maximum of a SP described by the diffusion Green function (27) appears at $t = t_m(x) = x^2/6\nu$ and its propagation velocity $V_m = [dt_m(x)/dx]^{-1}$ decreases as $1/\sqrt{x}$.

4. SHORT-DISTANCE BEHAVIOR OF HIGH-FREQUENCY SIGNALS

We consider here the propagation of WPs comprising a large number of oscillations, whose period is very small, compared to the envelope width Δ and the damping time λ. Such signals are solutions of the initial-boundary value problem (14) with

$$F(t) = H(t)\sin\omega t \tag{30}$$

where ω is the appropriate carrier frequency. We assume that the function $H(t)$ represents a temporal peak with the characteristic width $\Delta \gg 2\pi/\omega$ and that $H(t)$ vanishes at $t = 0$ and $t \to \infty$ Inserting (30) into (22) and performing two consecutive integrations by parts of the integral term of (22), we obtain the approximate solution as

$$\begin{aligned} u(x,t) = e^{-x/2\lambda c}\Bigg[\bigg(&H(\eta)[1+O(\varepsilon)] + \frac{\varepsilon}{2\omega}\frac{dH}{d\eta}\bigg)\sin\omega\eta \\ &- \varepsilon\left[1+O(\varepsilon)\right]H(\eta)\cos\omega\eta\Bigg], \\ \eta = t - \frac{x}{c}\ ,\ \ &\varepsilon = \varepsilon(x) = \frac{x}{4c\lambda^2\omega} \ll 1 \end{aligned} \tag{31}$$

This solution describes a WP that can be presented as

$$u = A(\eta, x)\sin[\omega\eta + \psi(\eta, x)] \tag{32}$$

where A is a slowly varying amplitude and ψ is a phase shift which is inconsequential for the subsequent analysis. The function $A(\eta, x)$ has the form

$$A(\eta, x) = e^{-x/2\lambda c}\left[H(\eta)[1 + O(\varepsilon(x))] + \frac{\varepsilon(x)}{2\omega}\frac{dH(\eta)}{d\eta}\right] \tag{33}$$

From (33) it follows that at $x = 0$ the amplitude function A becomes the envelope of the boundary signal, i.e., $A(\eta, 0) = H(t)$. For a fixed value of the coordinate x, the maximum of the envelope $A(\eta, x)$ in time is achieved at $\eta = \tilde{\eta} = \tilde{t} - x/c$ where $\tilde{\eta}$ is determined by

$$[\partial A(\eta, x)/\partial\eta]_{\eta=\tilde{\eta}(x)} = 0 \tag{34}$$

If the function $H(t)$ has a single maximum at $t = t_m > 0$ then for $\varepsilon(x) \ll 1$ the solution of (34) can be found by the perturbation method:

$$\tilde{\eta} = t_m + \varepsilon\eta_1 + O(\varepsilon^2) \tag{35}$$

and, therefore,

$$\left(\frac{\partial A}{\partial\eta}\right)_{\eta=\tilde{\eta}} = e^{-x/2\lambda c}\left[\left(\frac{d^2H}{dt^2}\right)_{t=t_m}\left(\eta_1 + \frac{1}{2\omega}\right)\varepsilon + O\left(\frac{H(t)}{\Delta}\varepsilon^2\right)\right] \tag{36}$$

Inserting (36) into (34) results in $\eta_1 = -1/2\omega$. Then (35) allows us to determine the instant $\tilde{t}(x)$, for which the temporal envelope reaches its maximal value

$$\tilde{t}(x) = t_m + \frac{x}{c}\left[1 - \frac{1}{8\lambda^2\omega^2}\left(1 + O\left(\frac{\Delta x}{c\lambda^2}\right)\right)\right] \tag{37}$$

From (37), the velocity of this temporal maximum is given by

$$\widetilde{V} = \frac{dx}{d\tilde{t}} \approx \frac{c}{1 - 1/(8\lambda^2\omega^2)} \approx c\left[1 + 1/\left(8\lambda^2\omega^2\right)\right] > c,$$
$$\left(x \ll c\lambda^2/\Delta\ ,\ 8\lambda^2\omega^2 \gg 1\right) \tag{38}$$

The WP solution presented by Eqs. (32) and (33) is valid for limited distances satisfying the condition $x \ll c\lambda^2/\Delta$. For such distances the dispersive effect is weak and it does not change the local wave number $k = \omega/c$ at the envelope maximum. Since in the case under consideration the wave number satisfies the condition

$$k^2 = \omega^2/c^2 \gg k_c^2 = 1/(4\lambda^2 c^2) \tag{39}$$

the group velocity W is real and it is given by the second of Eqs. (9). Expanding $W(k)$ in series in a small parameter k_c^2/k^2 results in

$$W(k) = \frac{c}{\sqrt{1 - k_c^2/k^2}} = c\left[1 + 1/\left(8\lambda^2\omega^2\right) + O[1/(\lambda^4\omega^4)]\right] \tag{40}$$

Comparing Eqs. (38), (40), we can see that for a high-frequency signal satisfying condition (39), the envelope's peak propagates with the group velocity $W(\omega/c)$. The latter exceeds the front velocity c. The same result can be obtained if we calculate the real part U_c of the group velocity for a real frequency from the first of Eqs. (13). An expansion of the difference $U_c - c$ in series in a small parameter $\alpha = 1/\lambda^2\omega^2$ results in a leading term $c/(8\lambda^2\omega^2)$, which has the same form as those given by Eqs. (38) and (40). Therefore, with the accuracy of the first order in α all three velocities $dx/d\tilde{t}$, W and U_c coincide.

Since the solution of the initial-boundary value problem (14) differs from zero only for $t > x/c$, the instant $\tilde{t}(x)$ never becomes less than the value x/c, i.e., the envelope's maximum may propagate over some finite distance with the velocity $W > c$ but it never overtakes the leading edge of the signal propagating with the velocity c. Actually (37) which determines the instant $\tilde{t}(x)$ can be used until the difference $\tilde{t} - t_m$ remains much less than λ^2/Δ. For long times the envelope's maximum decelerates, its velocity becomes less than c and tends to zero as $t \to \infty$. At this stage the diffusion effects dominate and lead to the lowering of both the amplitude and the local frequency.

5. SHORT-DISTANCE BEHAVIOR OF SOLITARY PULSES

In the present section we analyze an initial stage of propagation of SPs, for which the boundary function $F(t)$ has the form

$$F(t) = C\ \exp[\varphi(t)]\ ;\ \varphi(0) = -\infty\ ;\ \varphi(\infty) = -\infty\ ;\ C = const > 0 \tag{41}$$

It is assumed that the function $\varphi(t)$ has a single maximum at $t = t_*$, i.e.,

$$\varphi'(t_*) = 0 \ , \ \varphi''(t_*) < 0 \tag{42}$$

Consequently the function $F(t)$ vanishes at $t = 0$ and $t = \infty$ and it describes a pulse with a single maximum at $t = t_*$. In addition, we shall assume that the characteristic semi-width T of a pulse presented by (41) is much less than the damping time, i.e.,

$$T = \sqrt{2/|\varphi''(t_*)|} \ll \lambda \tag{43}$$

For short times and distances satisfying condition

$$x/c \leq t \ll \lambda \tag{44}$$

the argument z of the modified Bessel function is small and $I_1(z)$ can be replaced by

$$I_1(z) \approx \frac{z}{2}\left(1 + \frac{z^2}{8}\right) \tag{45}$$

The solution given by (22) can now be recast in an approximate form

$$u(x,t) \approx C\left[e^{\varphi(\eta)}q(x) + \frac{x}{8\lambda^2 c}\int\limits_{x/c}^{t} p(x,\tau)e^{\varphi(t-\tau)}d\tau\right],$$
$$\eta = t - x/c, \ q(x) = (1 - x/2\lambda c),$$
$$p(x,\tau) = (1 - \tau/2\lambda)\left[1 + \left(\tau^2 - x^2/c^2\right)/32\lambda^2\right] \tag{46}$$

Equation (46) describes the behavior of SPs at relatively small distances from the boundary $x = 0$, when both conditions (43), (44) are satisfied. The instant $\bar{t}(x)$ when the local maximum of the solution appears at the point x is determined from the condition $(\partial u/\partial t)_{t=\bar{t}} = 0$, which results in

$$e^{\varphi(\bar{\eta})}\varphi'(\bar{\eta})q(x) + \frac{x}{8\lambda^2 c}\left[p(x,\bar{t})e^{\varphi(0)} + \int\limits_{x/c}^{\bar{t}} p(x,\tau)e^{\varphi(\bar{t}-\tau)}\varphi'(\bar{t}-\tau)d\tau\right] = 0 \tag{47}$$

where $\bar{\eta} = \bar{t} - x/c$. The first term within the rectangular brackets vanishes since $\exp(\varphi(0)) = F(0) = 0$. For sufficiently small values of

$\bar{t}/\lambda$ and, therefore, for small $x/\lambda c$, we can replace in (47) each of the functions $q(x)$ and $p(x,\bar{t})$ by unity and retain only the factor linear in x before the rectangular brackets. Then the integral term in (47) can be calculated as

$$\int_{x/c}^{t} e^{\varphi(\bar{t}-\tau)}\varphi'(\bar{t}-\tau)d\tau = e^{\varphi(\bar{\eta})} \tag{48}$$

Thus, for sufficiently small values of x we obtain an asymptotic equation for the quantity $\bar{\eta} = \bar{\eta}(x)$

$$\varphi'(\bar{\eta}) + x/(8\lambda^2 c) = 0 \tag{49}$$

The solution of (49) at $x = 0$ is $\bar{\eta}(0) = t_*$. Therefore, for small distances x we can present $\bar{\eta}(x)$ as

$$\bar{\eta}(x) = t_* + \nu x + O(x^2) \tag{50}$$

This relation is similar to (35) that was obtained for the envelope's maximum. Inserting (50) into (49) results in

$$\nu = -\left[8\lambda^2 c\varphi''(t_*)\right]^{-1} = T^2/\left(16\lambda^2 c\right) \tag{51}$$

Therefore, an initial part of the trajectory of the solitary pulse maximum is approximated by a straight line in (x,t) plane

$$t = \bar{t}(x) = t_* + x/c + T^2 x/(16\lambda^2 c) \tag{52}$$

The maximum starts to propagate from the boundary with the velocity V that is determined by

$$V = \frac{dx}{d\bar{t}} = \frac{c}{1 + T^2/(16\lambda^2)} < c \tag{53}$$

This formula is valid if

$$x \ll c\lambda,\ t_* \ll \lambda,\ T^2/(16\lambda^2) \ll 1 \tag{54}$$

Formula (53) can be used for relatively small values of x and it gives an exact value of the temporal maximum velocity at the boundary

$x = 0$. The comparison of (54) with (38) shows that unlike the WP envelope maximum that starts to propagate from the boundary with a superluminal velocity W, the SP maximum starts with a subluminal velocity V. The second distinction lies in the fact that W equals the group velocity [see (40)], while the velocity of the SP maximum V is not related to the group velocity at all. It should be noted that small relative deviations of both velocities W and V from the speed of light c are linear in T^2/λ^2, where T is the minimal characteristic time of the boundary signal, i.e., T represents either the period of high-frequency oscillations or the characteristic semi-width of SP.

6. EXTENSION OF ANALYTICAL RESULTS TO GENERAL TE

The results presented in Sections 4 and 5 can be extended to the case of general telegrapher's equation (henceforth GTE)

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda}\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} - \omega_0^2 u \tag{55}$$

In comparison with TE given by (1), GTE contains the additional term $-\omega_0^2 u$. For a coaxial cable line this term appears when a finite shunt conductance per unit length Q is taken into account. For $Q > 0$ the parameters $1/\lambda$ and ω_0^2 are given by [see Stratton [9], (27), p.550]

$$\frac{1}{\lambda} = c^2(RC + LQ)\ ,\ \omega_0^2 = c^2 RQ \tag{56}$$

Using a well-known inequality

$$RC + LQ \geq 2\sqrt{RCLQ} = 2\sqrt{RQ}/c \tag{57}$$

results in

$$1/(4\lambda^2) \geq \omega_0^2 \tag{58}$$

It will be shown that condition (58) affects the qualitative behavior of WPs and solitary pulses in cable lines. According to Doetsch [12], the solution of the boundary-value problem for GTE is obtained from the solution of TE given by (22) when in the latter the second term is replaced by

$$\frac{x}{2\tilde{\lambda}c}\int\limits_{x/c}^{t} e^{-\tau/2\lambda}\frac{I_1\left[(2\tilde{\lambda})^{-1}\sqrt{\tau^2 - x^2/c^2}\right]}{\sqrt{\tau^2 - x^2/c^2}}F(t-\tau)d\tau \tag{59}$$

where $1/\tilde{\lambda}$ is defined as

$$\frac{1}{\tilde{\lambda}} = \sqrt{\frac{1}{\lambda^2} - 4\omega_0^2} \tag{60}$$

Therefore, in obtaining the analog of Eq. (35) for the velocity of the maximum of the wave packet envelope we just replace λ by $\tilde{\lambda}$ in the definition of $\varepsilon(x)$ given in Eq. (31). This leads to a modified formula (38), in which λ is replaced by $\tilde{\lambda}$. Since $1/\tilde{\lambda}$ is less than $1/\lambda$, taking account of a finite shunt conductance Q reduces the difference between the front velocity c and the velocity of the envelope's maximum. However, the latter remains larger than c for short distances x due to condition (58). The transition to GTE does not alter also the fact that for high-frequency oscillations the envelope's peak propagates at short distances with the real group velocity $W(k) > c$. We only should remember that now in Eqs. (39) and (40) λ is replaced by $\tilde{\lambda}$.

For a SP, the velocity of its temporal maximum V at small distances from the boundary $(x \ll c\tilde{\lambda})$ is determined by Eq. (53), in which λ is replaced by $\tilde{\lambda}$. Thus, the near-boundary value of V remains less than c and it is closer to c in comparison with the case of zero shunt conductance.

7. NUMERICAL RESULTS

The solution $u(x,t)$ of the initial-boundary value problem (14) for Eq. (1) was calculated numerically for high-frequency boundary signals having the form

$$F = F_0 \frac{t}{\Delta} \, \exp(-t/\Delta) \sin(\omega t) \ , \ F_0 = const \tag{61}$$

and for boundary signals representing SPs

$$F = F_0 \frac{t}{\theta} \, \exp(-t/\theta) \ , \ F_0 = const \tag{62}$$

Both solutions were calculated directly from the exact integral representation (22). The dimensionless variables $\xi = x/\lambda c$ and $\tau = t/\lambda$ were used. The software package Mathcad (version 7) was used to calculate numerically the Doetsch solution (22). The software used the adaptive Romberg method for integration [17] where the integration step is a parameter depending on the convergence of the results (integration error). The convergence parameter between every two integration steps $x = x_{n-1}$ and $x = x_n$ defined as $|\{J(x_n/c, t) - J(x_{n-1}/c, t)\}/F_0|$, where $J(x/c, t)$ is the integral term in formula (22), was chosen to be 10^{-12}. When calculating the approximate solution for small values of $t - x/c$, i.e., in the region where the argument of the modified Bessel function I_1 is very small, the approximate analytical formulas of Section 4 could be used. When the argument of I_1 could not be considered small, the interval of integration $[x/c, t]$ was divided into two sub-intervals, $D_1 \equiv [x/c,\ \sqrt{x^2/c^2 + 100\lambda^2}]$ and $D_2 \equiv [\sqrt{x^2/c^2 + 100\lambda^2},\ t]$. Within the second sub-interval the asymptotic expansion of the modified Bessel function $I_1(z)$ for large value of its argument z [15] was employed. In order to verify that Bessel function values were correct prior to the numerical integration in the sub-interval D_1, they were compared to the values obtained by MATLAB (version 5.1). The calculated values were found to be compatible. It has to be taken into consideration that the software Mathcad uses the Taylor expansion to describe the modified Bessel function $I_1(z)$, and therefore, an accuracy problem might arise if an insufficient number of terms is used. Various lengths of the Taylor expansion were tried until 40 terms were chosen to describe $I_1(z)$ for $z < 5$. For $z > 5$ only asymptotic expansion of $I_1(z)$ was employed for numerical integration in the sub-interval D_2.

When calculating the maximum value of the solution $u(\xi, \tau)$, where $\xi = x/c\lambda$, $\tau = t/\lambda$, 10^6 samples around the temporal maximum location were used. After finding the location of the temporal maximum τ_1 for a given ξ_1, the solution was computed for $\xi_1 + \Delta\xi$, and when the location of the next temporal maximum τ_2 was found, then the velocity of the temporal maximum was computed according to $\Delta\xi/(\tau_2 - \tau_1)$. This procedure was used in all calculations. In order to eliminate errors due to numerical noises, the interval $\Delta\xi$ was changed until results obtained were robust, i.e., not dependent on $\Delta\xi$. Typical values of $\Delta\xi$ were 10^{-3} to 10^{-5} and this was the reason for choosing 10^6 samples. When the WP propagation solution was calculated, some convergence problems arose for large arguments. This fact prevented the drawing of a full picture of the behavior of the maximum velocity, especially for large times. In the same way, calculations for the SP propagation were halted after $\xi = 18$, since we were confident with the results up to this point.

All results in this paper were checked by using an adaptive recursive Newton-Cotes eight panels rule integration method [17]. The numerical results obtained by the Newton-Cotes method and by the Romberg approach were found to be close.

The calculations of WP corresponding to the boundary signal (61) were performed for $\lambda/\Delta = 0.3$ and $\lambda\omega = 9.6$. The boundary signal is shown in Fig. 3. The velocity V_m/c of the absolute temporal maximum of the solution computed as a function of ξ is shown in Fig. 4. It is seen that near the boundary the maximum propagates with the superluminal velocity $(V_m/c > 1)$. The values of V_m/c become subluminal at distances x larger than $0.03L_a$ where $L_a = 2\lambda c$ is the characteristic absorption length. The calculations were halted at $x = L_a$ where the amplitude of the signal was significantly reduced.

Numerical calculations of SPs (62) performed for $\lambda/\theta = 4.5$ display that the temporal peak of the pulse begins to propagate from the boundary $x = 0$ with the velocity $V/c < 1$ in agreement with the analytical relation (53). Both the magnitude of the maximum and its velocity decrease with the distance x, while the trailing part of the pulse becomes progressively flatter (Figs. 5–8). At some critical distance $x = x_c$, which is close to five absorption lengths $(\xi_c = x_c/\lambda c \approx 10)$, the pulse profile representing the solution u as a function of time acquires an inflection point with a horizontal tangent. For $x > x_c$, a new maximum and minimum appear, so that the dependence of u of t has two maxima and one intermediate minimum. The magnitude of the first maximum decreases with time, while the magnitude of the second one increases and this new maximum becomes dominant. This intricate pulse form is observed until the first maximum and minimum approach one another and then disappear, as is seen in Fig. 8. This occurs at the distance x^* which is close to eight absorption lengths. For $x > x^*$ only one extremum of u (a maximum) exists. At large distances this maximum can be described by the Green function of the diffusion equation presented in (27). Its magnitude and velocity decrease with x, while the pulse width increases and becomes much larger than the width of the boundary pulse. At this long-distance stage of propagation“the memory”, i.e., the relation to the boundary signal features is totally lost.

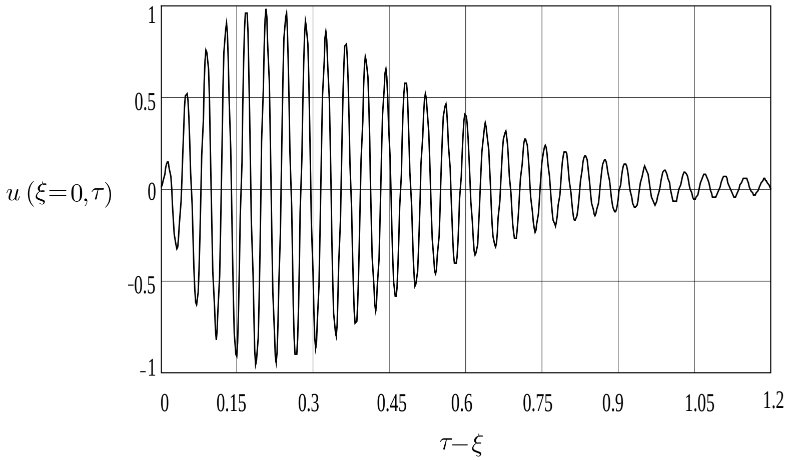

Figure 3. The WP $u(x = 0, t)$ given at the boundary of a half-space [formula (61)] versus the normalized time $\tau = t/\lambda$ for $\lambda/\Delta = 0.3$ and $\lambda\omega = 9.6$.

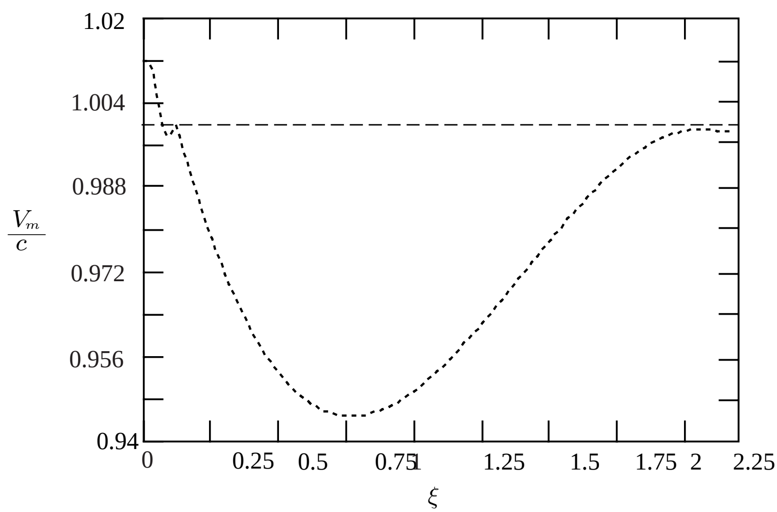

Figure 4. The normalized velocity V_m/c of the temporal maximum versus the normalized distance from the boundary $\xi = x/\lambda c$ for the solution $u(\xi, \tau)$ corresponding to the boundary signal presented in Fig. 3. The dots represent the locations where the velocity was computed.

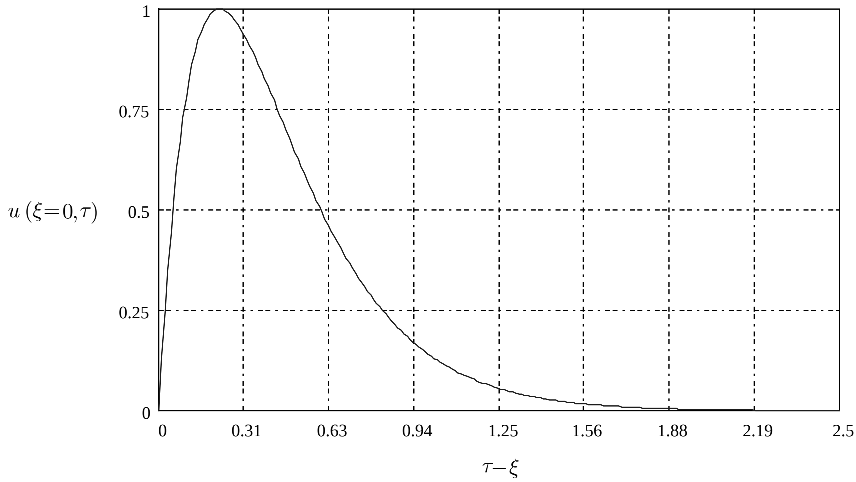

Figure 5. The SP $u(x = 0, t)$ given at the boundary of a half-space [formula (62)] versus the normalized time $\tau = t/\lambda$ for $\lambda/\theta = 4.5$.

8. DISCUSSION AND CONCLUSIONS

The objective of the present article is two-fold: to investigate thoroughly the spatial and temporal evolution of WPs and SPs propagating in a half-space and described by TE, and to demonstrate that the real group velocity appears as an important feature of solutions for large and small distances from the boundary.

As is shown in Section 3, the group velocity $W = \partial\omega/\partial k$ calculated for exponential solutions of the TE having the form $\exp[i(kx - \omega t)]$ is real in two cases: (a) when k is purely imaginary and, (b) when k is real and $|k| > 1/L_a$. Although WPs and single pulses are not elementary waves with constant values of k and ω, the dispersive properties of elementary waves are revealed when applying the saddle-point method to asymptotic solutions which can be characterized by the local wave number $k(x, t)$ and the local frequency $\omega(x, t)$. Thus, at large distances and long times when the asymptotic behavior of solutions is described by the Green function, slowly varying local characteristics $\omega(x, t) = \tilde{\omega}(x/t)$ and $k = k(\tilde{\omega}) = \tilde{k}(x/t)$ can be introduced. The quantity $\tilde{\omega}$ appears as a saddle point of the eikonal $\Sigma = k(\omega)x - \omega t$ in the complex plane ω. Both quantities $\tilde{\omega}$ and $\tilde{k}$ are purely imaginary and they can be associated with a local "group" of purely exponential waves in some vicinity of arbitrary point (x, t). Similar situation was considered in [16] for aperiodic solutions of the diffusion equation.

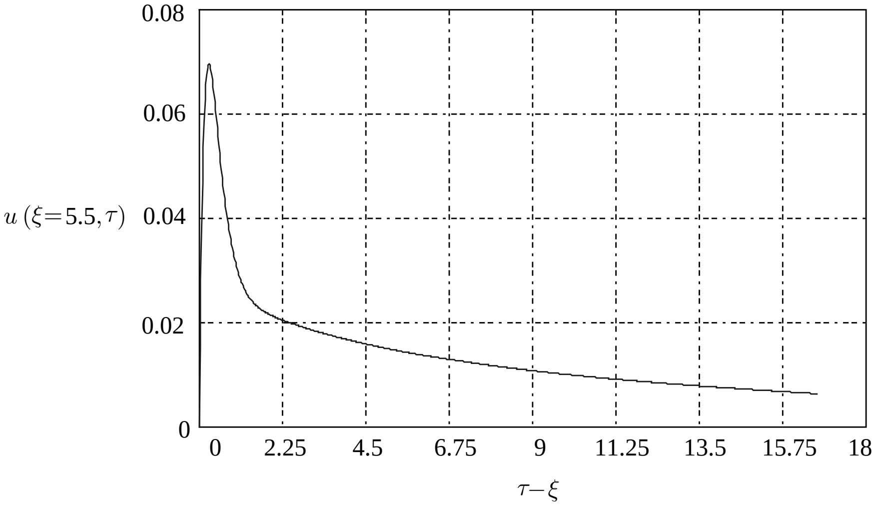

Figure 6. The SP $u(\xi, \tau)$ versus the time-dependent variable $\tau - \xi = (t - x/c)/\lambda$ for $\xi = x/\lambda c = 5.5$ and $\lambda/\theta = 4.5$.

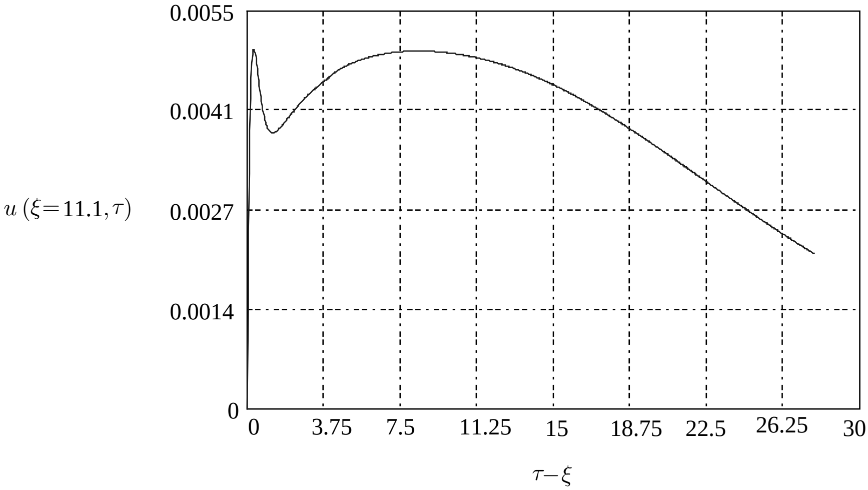

Figure 7. The SP $u(\xi, \tau)$ versus the time-dependent variable $\tau - \xi = (t - x/c)/\lambda$ for $\xi = x/\lambda c = 11.1$ and $\lambda/\theta = 4.5$.

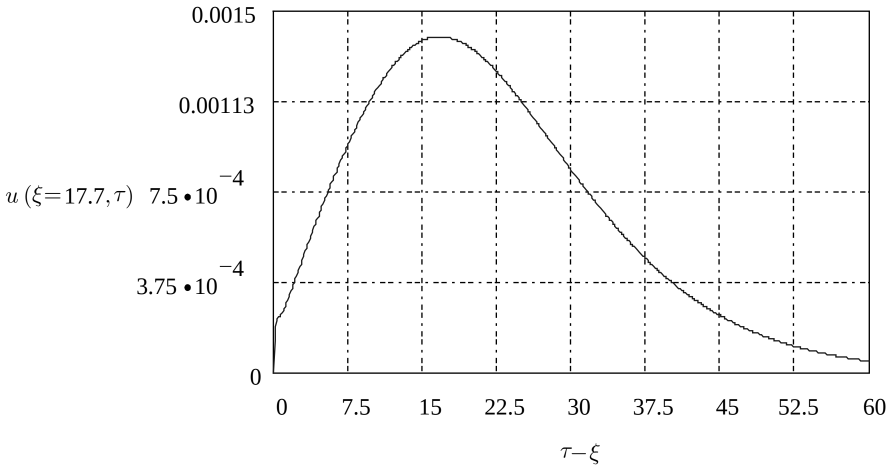

Figure 8. The SP $u(\xi, \tau)$ versus the time-dependent variable $\tau - \xi = (t - x/c)/\lambda$ for $\xi = x/\lambda c = 17.7$ and $\lambda/\theta = 4.5$.

Here the group velocity $W = x/t < c$ is not a velocity of propagation of a given real local frequency, as in the case of WPs in non-absorbing media [18], but is the velocity of propagation of a given local damping rate. This interpretation is in the agreement with the concept of real local group velocity for absorbing media developed in [11].

This chapter shows that the existence of the range of real wave numbers $|k| > 1/L_a$ in which the group velocity is superluminal $(W > c)$ cannot be ignored when studying causal solutions of the TE, and the superluminal group velocity, in the context in which it was presently introduced, should not be regarded as unphysical. When the group velocity in an absorbing medium $W(k)$ becomes real, it only has a kinematic meaning as a velocity of the interferential maximum for a WP concentrated near the central wave number k. For physical systems with combined wave damping and dispersion, governed by partial differential equations of hyperbolic type such as the TE or the GTE, the temporal envelope's maximum of a causal signal *may propagate with the local group velocity* $W > c$ for relatively short distances from the boundary. A proof of this statement is given in Sec. 4.

At a given point x the temporal envelope's maximum always appears after the instant $t = x/c$. Thus, there is no violation of relativity or causality: The information on switching on of the boundary signal propagates exactly with the speed of light c. As to the envelope's maxima that can be registered at two different points x_1 and $x_2 > x_1$, they appear after the instants x_1/c and x_2/c, respectively. Since the wave absorption leads to continuous reshaping of the envelope, the maximum which appears at $x = x_2$ is not a direct consequence of the maximum formed at $x = x_1$. Thus, the causality requirement does not impose any restriction on the velocity $\tilde{V}$ of the envelope's maximum. Since the envelope's maximum never overtakes the leading edge $x = ct$ of a causal signal, the instantaneous maximum velocity may be superluminal only at some finite distance from the boundary. For large distances $\tilde{V}$ becomes less than c. The deceleration of a superluminal motion with the transition through the speed of light was obtained in numerical calculations (see Fig. 3). In these calculations the near-boundary values of $\tilde{V}$ exceed the speed of light only by one percent due to the large value of the parameter $\omega\lambda$. The quantity $\tilde{V}$ increases when the parameter $\omega\lambda$ is reduced. However, the reduction of this parameter increases the damping rate to the frequency ratio at $x > 0$. As a result, the oscillations survive only within a very narrow boundary layer. To explain this behavior we can consider Eqs. (8) and (9) for real $k^2 > k_c^2$. Then we obtain

$$|Im\omega/Re\omega| = \sqrt{W^2/c^2 - 1} \tag{63}$$

Thus, formula (63) shows that for $W/c > 1.1$ the ratio $|Im\omega/Re\omega|$ is larger than 0.46. Since in this case the damping time $1/|Im\omega|$ becomes at least three times less than the period of oscillations $2\pi/|Re\omega|$, the concept of a slow varying envelope fails. Similar results are obtained when considering the behavior of the real part $U_c(\omega)$ of the group velocity calculated for real ω [see the first equation in (13) and Fig.1]. The maximal value of the ratio U_c/c equals 1.066 and it is reached at $\omega\lambda = 0.796$. However, for this value of $\omega\lambda$ the wave length $2\pi/k_r$ is more than three times larger than the damping length $1/k_i$ and we can see again that the concept of slowly varying envelope of oscillations (now in space) fails. Thus, the velocity of an interferential maximum presented by any of the quantities $W(\omega/c)$ or $U_c(\omega)$ may exceed the speed of light only slightly and this happens only in the domain where the short-wave or high-frequency WPs are well defined, i.e., when the WP's envelope is varying slow. This is confirmed also by the analysis of the solution for a half-space performed in Section 4. The reduction of the parameter $\omega\lambda$ without changes of the effective number of oscillations at $x = 0$, i.e., when keeping $\omega\Delta = const$, decreases the ratio λ/Δ. This leads to the lowering of the relative size x/L_a of the nearby

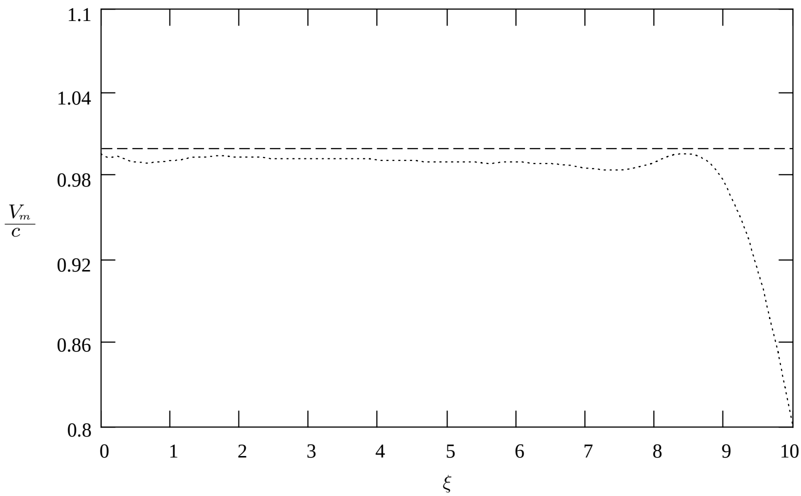

Figure 9. The velocity V of the SP maximum normalized to speed of light c versus the normalized distance from boundary $\xi = x/\lambda c$ for the solution $u(\xi, \tau)$ corresponding to the boundary signal presented in Fig. 5. The dots represent the locations where the velocity was computed.

boundary domain in which the condition $\tilde{V} > c$ can be guaranteed [see the second inequality in formulas (38)].

The behavior of aperiodic SPs whose characteristic width at the boundary is less than the damping time $(T \ll \lambda)$ is interesting. At short distances which are less than the absorption length $(x \ll L_a)$, the pulse maximum propagates with the velocity V that is close to but less than c (Fig. 9). The difference $c - V$ increases with the increase of the ratio T/λ. Since for very large distances $(x \gg L_a)$ and long times $(t - x/c \gg \lambda)$ the solution should evolve to a diffusion-type pulse described by the Green function (27), one might think at a first glance, that the solitary boundary pulse would be extended in time and decrease its magnitude to approach the diffusion-like form without any peculiarities. Instead, however, a strong reshaping of the pulse occurs within several absorption lengths, as is seen from Figs. 5–7. The transition from the wave-like mode to the diffusion-like mode of propagation is associated with the formation of a new temporal maximum, while the first maximum totally disappears at sufficiently large distances. Such a behavior is a consequence of competition between two terms representing the exact solution of TE given by (22). Each of these terms has a temporal maximum, while the maximum of the first, local, term appears first and dominates at short distances where the magnitude of the second, integral, term is small. For large distances the first term is suppressed by absorption and the integral term becomes dominant.

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